BOUNDARY ELEMENT METHODS FOR VIBRATION PROBLEMS

by

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Code 542
MY RESEARCH IN LAST FEW YEARS

Noise reduction – passive approaches

Helmholtz Resonator

Noisy Structure

Stiffeners

Dynamic analysis

Acoustic analysis
MOST RECENT RESEARCH

Parallel optimization algorithms

![Graph showing speedup vs. number of processors]

- bracketing
- interval reduction
- total line search
- linear speedup (reference)
OBJECTIVES AT NASA: SUMMER 2003

Tutorial ON Boundary Element Analysis ("BEM")

Computer codes (for tutorial and problem solving)

Study the potential for BEM in Vibration at mid-frequency levels
  i.e. fill the ‘gap’ between FEA and SEA in view of:
    Low freq – FEA is very good
    High freq -- SEA seems adequate (although
detailed response is not obtainable)

Assess the state of the art in BEM for vibration analysis
BEM -- SIMPLE 1-D EXAMPLE

\[ \frac{d^2u}{dx^2} + x = 0, \quad u(0) = 0, \quad u(1) = 0 \]

\[ \int_0^1 w \left( \frac{d^2u}{dx^2} + x \right) dx = 0 \]

Integrate by parts twice:

\[ w u' \big|_0^1 - u w' \big|_0^1 + \int_0^1 u \frac{d^2w}{dx^2} dx + \int_0^1 w x dx = 0 \]

Choose \( w \) such that

\[ \frac{d^2w}{dx^2} = -\delta(x - x_i) \]

‘Fundamental Solution’:

\[ w(x) = - \text{step}(x - x_i) \ (x - x_i) \]
SIMPLE EXAMPLE – cont’d

Then

\[
\begin{align*}
    u(x_i) &= w \left. u \right|_0^1 - u \left. w \right|_0^1 + \int_0^1 w \cdot x \, dx \\
    &= \underbrace{\text{boundary terms}}_{\text{source , and } w(x) \text{ depends on source point } x_i} + \underbrace{\text{domain term}}_{\text{source , and } w(x) \text{ depends on source point } x_i}
\end{align*}
\]

\[x_i = \text{‘source ’}, \text{ and } w(x) \text{ depends on source point } x_i\]

Choose source \(x_i = 0^+ = 0 + \varepsilon\). Thus \(w(0) = 0\) and remains zero as \(\varepsilon \to 0\). \(w(1) = -1, \ w'(1)= -1\), body force term \(= -1/3\), and we get

\[u'(1) = -1/3\]

We can also recover the solution for all interior points from Eq. (*) : \(u(x_i) = (x_i - x_i^3)/6\)

Note: the differential (equilibrium) is exactly satisfied within the domain in BEM – while boundary conditions are approximately satisfied, here, the boundary is only wo points, so BEM gives exact solution
AXIAL VIBRATION

\[ \ddot{u} - c^2 \ u'' = p(x,t) \]

Harmonic loading:  \[ u'' + k^2 \ u = - p(x,t) / c^2 \]

\[ k = \frac{\omega}{c}, \quad c = \sqrt{E/\rho} \]

\[ \int_0^L \left( u'' + k^2 \ u + p / c^2 \right) w \ dx = 0 \]

Integrate by parts twice,

\[ u' \ w|_0^L - u \ w'|_0^L + \int_0^L u (w'' + k^2 \ w) \ dx + \int_0^L p w / c^2 \ dx \]

Fundamental solution \( w \):

\[ w'' + k^2 \ w = - \delta(x, x_i) \]

\[ w = \frac{-1}{k} \sin k(x - x_i) \ \text{step}(x, x_i) \Rightarrow \]

\[ u(x_i) = u' \ w|_0^L - u \ w'|_0^L + \int_0^L p w / c^2 \ dx \quad (\ast) \]
Example

\[
\begin{array}{c}
\rho A = 1 \\
u(0) = 0 \\
du/dx (L) = F/c^2
\end{array}
\]

From Eq. (*), tip displacement, \( u_L = \frac{F}{k c^2} \tan(kL) \)

This is exact, since boundary conditions are exactly satisfied.

Resonances occur when \( \cos(kL) = 0 \), or \( kL = \pi/2, 3\pi/2, ... \)

Eq. (*) then gives

\[
u(x_i) = \frac{F}{k c^2} \{\tan(kL) \cos(k(L - x_i)) - \sin(k(L - x_i))\}
\]

Timoshenko’s formula:

\[
u_L = \frac{2F}{c^2 L} \sum_{i=1,3,5,...} \frac{1}{\left(\frac{i^2 \pi^2}{4 L^2} - k^2\right)}
\]

Consider 3 modes in the expansion (FEA needs about 3 elements for each half sine wave) - red color in fig:
Mode 4 = blue color
BEAM VIBRATION

\[ \frac{\partial^4 v}{\partial x^4} - k^2 v = \frac{p}{\rho Ac^2} \]

\[ \frac{\partial^4 w}{\partial x^4} - k^2 w = -\delta(x, x_i) \quad \Rightarrow \]

\[ w = -\text{step}(x, x_i) \frac{1}{2k^{3/2}} \left\{ \sinh \left[ \sqrt{k} (x - x_i) \right] - \sin \left[ \sqrt{k} (x - x_i) \right] \right\} \]

\[ v(x_i) = \ldots \]

But now, also need to differentiate \( v \) wrt \( x_i \) to get additional equation(s) for solution
STATIC PLATES BY DIRECT BEM

\[ \nabla^4 w = \frac{p}{D} \quad \text{Kirchhoff theory for thin plates} \]

\[ \iint_A (w \nabla^4 w^* - w^* \nabla^4 w) \, dA = \iint_S (w^* V - V^* w - \theta^* M + M^* \theta) \, dS \]

+ corner terms

Choose the ‘concentrated force’ fundamental solution \( w^* \) such that

\[ \nabla^4 w^*(x_i) = \delta(x, x_i) \]

which gives

\[ w^* = \frac{1}{4 \pi D} r^2 \ln r \]

Thus

\[ c_i w(x_i) = \frac{1}{D} \iint_A p \, w^* \, dA + \iint_S (w^* V - V^* w - \theta^* M + M^* \theta) \, dS \]

\[ + \sum_{k=1}^{K'} (w^* F - F^* w) \]

where \( c_i \) equals 0.5 if \( x_i \) is on smooth \( S \), and equals 1 if interior.
A second equation is obtainable as:

\[
\begin{align*}
&c_{\xi} \frac{\partial w}{\partial \xi} + c_{\eta} \frac{\partial w}{\partial \eta} = \frac{1}{D} \int_{A} p \ w^* \ dA + \int_{S} \left( w^* \ V - V^* \ (w - w_i) - \theta^* \ M + M^* \ \theta \right) dS \\
&\quad + \sum_{k=1}^{K} \left( w^* \ F - F^* \ (w - w_i) \right)
\end{align*}
\]

where the ‘concentrated moment’ fundamental solution \( w^* \) is given by

\[
w^* = \frac{1}{2 \pi D} r \ \ln r \ \cos(\alpha)
\]

Expressions for \( V^* \), \( V'^* \) etc are given in the literature.

Note: Corners (eg. rectangular plates) require special attention.

Distributed load requires domain integration.

Here, a function \( s \) is defined such that

\[
\nabla^2 s = w^*, \ \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \ \frac{\partial}{\partial r}
\]

and then converting to a contour integral as

\[
p \int_{A} w^* \ dA = p \int_{S} \frac{\partial s}{\partial r} \cos(\beta) \ dS
\]

where \( \beta = \) angle between \( r \) and \( n \). Similarly for \( w'^* \).
BEM WITH CONSTANT ELEMENTS: CONTOUR INTEGRATION

Thus, \( \int_{e} V^{*} w \, dS = w_j \int_{e} V^{*} \, dS \) which is evaluated using gauss quadrature

Linear elements will provide more accuracy

Finally: \( \mathbf{A} \mathbf{x} = \mathbf{b} \)

\( \mathbf{A} \) is square, unsymmetric, dense, \( \text{dim } 2N+3K \)
where \( N = \) no. of regular nodes, \( K = \) no of corner nodes

Clamped Plate: \( \mathbf{x} = [\text{shear, moment}] \) per unit length
Simply-Supported Plate: \( \mathbf{x} = [\text{slope, shear}] \)
DYNAMIC ANALYSIS OF PLATES

\[ \nabla^4 w + \frac{\rho h}{D} \ddot{w} = \frac{p}{D} \]

For general transient loading, apply Laplace transform and then use BEM. Here, we assume harmonic loading. Thus,

\[ w(x,t) = w(x) e^{i\omega t}, \quad w(x) = \text{amplitude} \]

\[ \nabla^4 w - k^2 w = \frac{p}{D}, \quad k = \frac{\omega}{c} \]

Choose the ‘concentrated force’ fundamental solution \( w^* \) such that

\[ \nabla^4 w^*(x_i) - k^2 w = -\delta(x,x_i) \]

\[ w^* = \frac{i}{8k} \left[ H_0^{(2)}(\sqrt{k} \ r) + H_0^{(1)}(i\sqrt{k} \ r) \right] \]

The ‘concentrated moment’ fundamental solution \( w'^* \) is given by

\[ w'^* = \cos(\alpha) \left[ i c_1 J_1(\sqrt{k} \ r) + c_1 Y_1(\sqrt{k} \ r) + \right. \]

\[ \left. c_2 K_1(\sqrt{k} \ r) \right] \]
CORNER EFFECTS

When source point is a corner, two concentrated moment fundamental solutions, and corresponding equations, need to be written. There are a total of three unknowns at each corner.

When a plate is loaded, corners tend to curl up – corner reactions keep these clamped.

Details are omitted here for brevity.
HANDLING DOMAIN (PRESSURE) LOADS IN DYNAMICS

\[ \frac{1}{D} \iint_A p w^* \, dA , \quad \frac{1}{D} \iint_A p w^* \, dA \]

**Approach 1:** Create a domain mesh, and integrate (careful mesh design, and integration are necessary for accuracy)

**Approach 2:** Express \( p w^* = \sum_{i=1}^{m} a_i \psi_i \), where \( a_i \) are determined through regression, and each of the known functions \( \psi_i \) can be written as \( \nabla^2 s_i \). Then, the Divergence theorem allows us to convert the domain integral to a contour integral.
DIFFICULTIES IN BEM

1. Fundamental solutions are hard to derive for complicated differential equations

2. Domain integration requires special care

3. Numerical integration involving bessel functions need special care
BEM WITH MULTIPLE RECIPROCITY  
(BEM-MRM)

$$\nabla^4 w - k^2 w = \frac{P}{D}$$

Choose the ‘concentrated force’ fundamental solution $w^*$ such that

$$\nabla^4 w^*(x_i) - k^2 w = -\delta(x, x_i)$$

Choose $w^*$ as the static fundamental solution:

$$\nabla^4 w^*(x_i) = \delta(x, x_i)$$

which gives  

$$w^* = \frac{1}{4\pi D} r^2 \ln r$$

We then have (ignoring corner terms):

$$c_i w(x_i) = k^2 \iint_A w w^*_s \, dA +$$

$$\frac{1}{D} \iint_A p w^*_s \, dA + \int_S \left( w^*_s V - V_s^* w - \theta_s^* M + M_s^* \theta \right) dS$$

Let $\nabla^4 s_1^* = w^*_s$, $\nabla^4 s_2^* = s_1^*$, $\nabla^4 s_3^* = s_2^*$ etc

and repeatedly use, with $s = s_i$,

$$\iint_A (w \nabla^4 s^* - s^* \nabla^4 w) dA = \int_S \left( s^* V - V_s^* w - \theta_s^* M + M_s^* \theta \right) dS$$
to obtain an expression for the displacement

\[
c_i w(x_i) = \frac{1}{D} \int\int_A p \hat{w}^* \, dA + \int_S \left( \hat{w}^* V - \hat{V}^* w - \hat{\theta}^* M + \hat{\dot{M}}^* \theta \right) dS + \sum_{k=1}^{K'} \left( \hat{w}^* F - \hat{F}^* w \right)
\]

where

\[
\hat{w}^* = w_s^* + \sum_{i=1}^{p} k^{2i} s_i^* , \quad \hat{V}^* = V_s^* + \sum_{i=1}^{p} k^{2i} V_{s_i}^*
\]

etc, along with a similar equation for the slope.
Advantages of BEM_MRM

Simple fundamental solutions regardless of complexity of differential equation

All domain integrals can be easily converted to contour integrals

Claimed that number of terms, $p$, are small for convergence – is this true for all frequencies?

Integrals independent of $k$ can be done once and stored within the frequency loop
COMPUTER CODE

Circular plate (done), Rectangular plate (in progress)
Concentrated load, Constant boundary element
Damping: \( E = E_0 (1 + i \eta) \)

Input: pressure as SPL (converted to point load)

Outputs:

Center displacement Vs frequency

\[
PSD_b = \frac{1}{N_b} \sum_{n} [acc(n)]^2 ,
\]

where \( N_b = (f_u - f_{\ell}) \),
\( f_u \) and \( f_{\ell} \) = upper \& lower freq in band, at 1Hz inc.
\( acc = \) acceleration in g's
\( acc(n) = \left( -\frac{\omega_n^2 w_n}{g} \right) \), \( \omega_n = \frac{f_n}{2\pi} \)

\[
GRMS^2 = \frac{1}{2} \sum_{n=1}^{\infty} [acc(n)]^2
\]

Validation: with Timoshenko’s formulas and with Ansys
VALIDATION EXAMPLE

Clamped Circular Steel Plate, R=1m, $\eta=.03$
Concentrated load = 1 N
ANSYS RESULT

Circular plate concentrated
SAMPLE PROBLEM

Clamped Circular Steel Plate
1 m radius, 1 cm thick, $\eta = .01$

Pressure corresponds to about 50 Pa
22 1/3-octave bands, ranging from 16 Hz to 2000 Hz
CENTER DISPLACEMENT
Vs HZ

Circular Plate
MEAN ACCELERATION IN G’S IN EACH 1/3RD OCTAVE BAND

Circular Plate
PSD IN EACH 1/3\textsuperscript{RD} OCTAVE BAND
DRAWBACK IN MASS MATRIX CALCULATIONS IN FINITE ELEMENTS

\[ u = N \mathbf{q} \]

\( u \) = displacement field in element \( e \)
\( \mathbf{q} \) = nodal displacements of \( e \)
\( N \) = shape functions – polynomials

Mass matrix:
\[
\mathbf{m} = \frac{1}{2} \int_{e} \mathbf{N}^T \mathbf{N} \, dV
\]

\( \mathbf{M} \) is assembled from each \( \mathbf{m} \)

Choice of \( N \) critical in dynamics, since inertial force equals \( \omega^2 \mathbf{M} \), and solution is obtained from
\[
[\mathbf{K} - \omega^2 \mathbf{M}]^{-1} \mathbf{F}
\]

Comment: for large \( \omega \), polynomial shape functions are inadequate

Also, shear effects must be properly captured
A FEW KEY REFERENCES

Book: “Boundary elements – an introductory course”, 2nd ed, C.A. Brebbia and J. Dominguez

Paper: “Boundary integral equations for bending of thin plates”, Morris Stern


Papers on BEM-MRM: see J. Sladek.
SUMMARY OF WORK COMPLETED

Tutorial has been developed for BEM in vibration
-- rods, beams, Laplace eq., plate bending

Computer code for circular plate vibrations
clamped or simply-supported
validated with Closed-form solutions and Ansys
SPL input, PSD and and other metrics output

State of the art for vibration analysis has been assessed

Rectangular (and other geometries) on-going

Research issues identified (see next slide)
FUTURE WORK IN APPLYING BEM FOR PLATE VIBRATIONS at MID-FREQUENCIES

Accurate and efficient numerical integration of body force terms

Develop BEM-MRM and compare with ‘standard’ BEM; more validation

Include shear effects (BEM-MRM is attractive)

Generalize geometries (any planar shape, thick plates, shells)

Compare BEM, FEM, Experiment on a specific problem;

Spectral element methods

Other research: (shock loading, optimization)