We summarize the second derivation in the text – the one that uses a *differential control volume*. First, we approximate the mass flow rate into or out of each of the six surfaces of the control volume, using *Taylor series expansions* around the center point, where the velocity components and density are \( u, v, w, \) and \( \rho \). For example, at the right face,

\[
(\rho u)_{\text{center of right face}} = \rho u + \frac{\partial (\rho u)}{\partial x} \frac{dx}{2} + \frac{1}{2!} \frac{\partial^2 (\rho u)}{\partial x^2} \left( \frac{dx}{2} \right)^2 + \cdots
\]  

(9–6)

The mass flow rate through each face is equal to \( \rho \) times the normal component of velocity through the face times the area of the face. We show the mass flow rate through all six faces in the diagram below (Figure 9-5 in the text):

Next, we add up all the mass flow rates through all six faces of the control volume in order to generate the general (unsteady, incompressible) *continuity equation*:
We plug these into the integral conservation of mass equation for our control volume:

\[
\int_{CV} \frac{\partial \rho}{\partial t} \, dV = \sum_{\text{in}} \dot{m} - \sum_{\text{out}} \dot{m}
\]  

(9-2)

This term is approximated at the center of the tiny control volume, i.e.,

\[
\int_{CV} \frac{\partial \rho}{\partial t} \, dV \approx \frac{\partial \rho}{\partial t} \, dx \, dy \, dz
\]

The conservation of mass equation (Eq. 9-2) thus becomes

\[
\frac{\partial \rho}{\partial t} \, dx \, dy \, dz = - \frac{\partial (\rho u)}{\partial x} \, dx \, dy \, dz - \frac{\partial (\rho v)}{\partial y} \, dx \, dy \, dz - \frac{\partial (\rho w)}{\partial z} \, dx \, dy \, dz
\]

Dividing through by the volume of the control volume, \(dx \, dy \, dz\), yields

**Continuity equation in Cartesian coordinates:**

\[
\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} + \frac{\partial (\rho w)}{\partial z} = 0
\]  

(9-8)

Finally, we apply the definition of the **divergence** of a vector, i.e.,

\[
\nabla \cdot \vec{G} = \frac{\partial G}{\partial x} + \frac{\partial G}{\partial y} + \frac{\partial G}{\partial z} \quad \text{where} \quad \nabla = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \quad \text{and} \quad \vec{G} = (G_x, G_y, G_z)
\]

Letting \(\vec{G} = \rho \vec{V}\) in the above equation, where \(\vec{V} = (u, v, w)\), Eq. 9-8 is re-written as

**Continuity equation:**

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V}) = 0
\]  

(9-5)