Smart Mater. Struct. 3 (1994) 47–58. Printed in the UK

Fatigue damage control of mechanical systems

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Received 26 August 1993, in final form 5 November 1993

Abstract. This paper presents the concept and architecture of a fatigue damage control system for mechanical structures. In contrast to the conventional cycle-based approach, fatigue damage is represented via non-linear differential equations with respect to time in the state-variable setting. This damage model is compatible with the dynamic model of the plant, i.e. the process under operation and control, and the instantaneous damage rate depends on the current level of accumulated damage. The objective here is to achieve an optimized trade-off between dynamic performance and structural durability of the plant. This interdisciplinary effort requires augmentation of the system-theoretic techniques for decision making and control with governing equations and inequality constraints representing the fatigue damage properties of structural materials. The major challenge in the reported work is to characterize the fatigue damage generation process in mechanical structures and then utilize this information for synthesizing algorithms of performance optimization, robust control and risk assessment for plant operation.

1. Introduction

Performance, mission objectives, service life, and maintenance cost must be taken into consideration for operation and control of complex mechanical systems such as advanced aircraft, spacecraft, and power plants. However, the current state-of-the-art of control systems synthesis focuses on improving dynamic performance and diagnostic capabilities under the constraints that often do not adequately represent the material properties of the critical plant components. The reason is that the process planning and design is traditionally based upon the assumption of invariant characteristics of conventional materials. In view of high performance requirements and the availability of improved materials that may have significantly different damage characteristics relative to conventional materials, the lack of appropriate knowledge about the properties of these materials will lead to either of the following:

- less than achievable performance due to overly conservative design; or
- over-straining of the structure leading to unexpected failures and a drastic reduction of the useful life span.

For example, reusable rocket engines present a significantly different problem in contrast to expendable propulsion systems that are designed on the basis of minimization of weight and acquisition cost under the constraint of specified system reliability. In reusable rocket engines, multiple start–stop cycles cause large thermal strains; steady-state stresses generate inelastic strains; and dynamic loads induce high cyclic strains leading to fatigue failures. The original design goal of the Space Shuttle Main Engine (SSME) was specified for 55 flights before any major maintenance, but the current practice is to dissemble the engine after each flight for maintenance (Lorenzo and Merrill 1991a). Another example is the design modification of the F-18 aircraft as a result of conversion of the airframe structure. In this case a major goal of the vehicle control systems redesign should be to achieve a trade-off between flight manoeuverability and durability of the critical components (Noll *et al* 1991).

Although a significant amount of research has been conducted in each of the individual areas of control and failure diagnostics in mechanical structures and analysis and prediction of materials damage, integration of these two disciplines has not received much attention. In view of the integrated structural and flight control of advanced aircraft, Noll *et al* (1991) have pointed out the need for research in the fields of active control technology and structural integrity, specifically fatigue life analysis. The direct benefits of the interdisciplinary approach are as follows:

- extension of the service life of the controlled process;
- increase of the mean time between major maintenance actions;

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- reduction of risk in the integrated control–structure–materials system design.

The key idea of the work reported in this paper is that it is possible to achieve a substantial improvement in the service life by a small reduction in the system dynamic performance. The objective is to achieve an optimized trade-off between dynamic performance and structural durability of the plant. To this effect, a dynamic model of fatigue damage in mechanical structures has been formulated in the continuous-time setting instead of the conventional cycle-based approach. The fatigue damage is represented via non-linear differential equations with respect to time in the state-variable setting. This damage model is compatible with the dynamic model of the plant, i.e., the process under operation and control, and the instantaneous damage rate depends on the current level of accumulated damage. The major challenge in the reported work is to characterize the fatigue damage generation process in mechanical structures and then utilize this information for synthesizing algorithms of performance optimization, robust control, and risk assessment for plant operation.

This paper is organized in four sections and two appendices. A framework of the damage-mitigating control system is proposed in section 2. A model of fatigue damage dynamics is derived in the continuous-time setting in section 3. A non-linear fatigue damage model is then developed as a modification of the linear model. Finally the paper is summarized and concluded in section 4. A discussion on how the predicted damage can be used for failure prognosis and condition-based risk analysis is presented in appendix A. An alternative approach to fatigue damage modeling is outlined in appendix B.

2. The concept of damage-mitigating control

The damage-mitigating control system, also referred to as the life extending control system (Lorenzo and Merrill 1991b), is intended to function independently or as an integral part of a hierarchically structured control system. Figure 1 shows a conceptual view of the damage prediction system which is an essential ingredient of the proposed damage-mitigating control system. The plant model is a finite-dimensional state-space representation of the system dynamics (e.g., thermal–hydraulic dynamics of the space shuttle main engine, or propulsion and aerodynamics of an aircraft). The plant states are inputs to the structural model which in turn generate the necessary information for the damage model. The output of the structural model is the load vector which may consist of (time-dependent) variables such as stress, strain and temperature at the critical point(s) of the structure. The damage model is a continuous-time (instead of being a cycle-based) representation of life prediction such that it can be incorporated within the control system model in the state-variable setting. The objective is to include, within the control system, the effects of damage rate

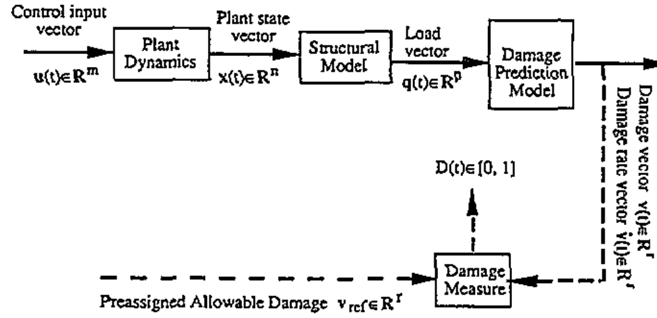

Figure 1. The damage prediction system.

and accumulated damage at the critical point(s) of the structure which may be subjected to time-dependent load. The damage state vector $v(t)$ could indicate the level of micro-cracking, macroscopic crack length, wear, creep, density of slip bands, etc at one or more critical points, and its time derivative $\dot{v}(t)$ indicates how the instantaneous load is affecting the structural components. The overall damage $D(t)$ is a scalar measure of the combined damage at the critical point(s) resulting from (possibly) different effects (e.g., fatigue and creep) relative to the preassigned allowable level v_{ref} of the damage vector. Although $D(t)$ may not directly enter the feedback or feedforward control of the plant, it can provide useful information for intelligent decision making such as damage prognosis and risk analysis.

The plant and damage dynamics in figure 1 are modeled by non-linear (and possibly time-varying) differential equations which must satisfy the local Lipschitz condition (Vidyasagar 1992) within the domain of the plant operating range. The structural model in figure 1 consists of solutions of structural dynamic (typically finite-element) equations representing the (mechanical and thermal) load conditions. A general structure of the plant and damage dynamics and their constraints is represented in the deterministic setting as follows.

Task period: starting time t_0 to final time t_f

Plant dynamics:

$$\dot{x} \equiv \frac{dx}{dt} = f(x(t), u(t), t); \qquad x(t_0) = x_0 \tag{1}$$

Damage dynamics:

$$\dot{v} \equiv \frac{dv}{dt} = h(v(t), q(x,t), t); \qquad v(t_0) = v_0; \qquad h \geqslant 0 \quad \forall t \tag{2}$$

Damage measure:

$$D(t) = \xi(v(t), v_{ref}) \qquad \text{and} \qquad D(t) \in [0, 1] \tag{3}$$

Damage rate tolerance:

$$0 \leqslant h(v(t), q(x,t), t) < \beta(t) \quad \forall t \in [t_0, t_f] \tag{4}$$

Accumulated damage tolerance:

$$[v(t_f) - v(t_0)] < \Gamma \tag{5}$$

where $x \in \boldsymbol{R}^n$ is the plant state vector; $u \in \boldsymbol{R}^m$ is the control input vector; $v \in \boldsymbol{R}^r$ is the damage state vector; $v_{ref} \in \boldsymbol{R}^r$ is the preassigned limit for the damage state vector; $\beta \in \mathbf{R}^r$ and $\Gamma \in \mathbf{R}^r$ are specified tolerances for the damage rate and accumulated damage, respectively; $q \in \mathbf{R}^p$ is the load vector; and $D \in [0, 1]$ is a scalar measure of the accumulated damage.

The state-variable representation of the fatigue damage model in equation (2) allows the instantaneous damage rate $\dot{v}(t)$ to be dependent on the current level $v(t)$ of accumulated damage. The physical interpretation of the above statement is that a given test specimen or a plant component, under identical stress–strain hysteresis, shall have different damage rates for different initial damage. For example, if the initial crack length is 100 μm, the crack propagation rate will be different from that for a crack length of 20 μm under identical stress excursions.

The vector differential equations (1) and (2) become stochastic if the randomness of plant and material parameters is included in the models, or if the plant is excited by discrete events occurring at random instants of time (Sobczyk and Spencer 1992). Failure prognosis and risk assessment based on on-line damage prediction requires stochastic modeling of the damage dynamics. Although the stochastic aspect of damage-mitigating control is not addressed in this paper, the basic issues in failure prognosis and probabilistic risk assessment are outlined in appendix A.

2.1. The open-loop control policy

Given an initial condition, the open-loop control policy is obtained via non-linear programming (Luenberger 1984) by minimizing a specified cost functional under the *prescribed constraints* of damage rate and accumulated damage. The objective is to minimize a cost functional J (which includes plant state, damage rate, and control input vectors) without violating the prescribed upper bounds of the damage rate and the accumulated damage. The cost functional J is to be chosen in an appropriate form representing a trade-off between the system performance and the damage. The optimization problem is then formulated as follows:

Minimize:

$$J = \sum_{k=0}^{N-1} J_k(\tilde{x}_k, \dot{v}_k, \tilde{u}_k) \qquad (6)$$

subject to:

$$0 \leqslant h(v_k, q(x_k), k) < \beta(k) \text{ and } (v_N - v_0) < \Gamma \quad \text{for } k = 1, 2, 3, \ldots, N \qquad (7)$$

where $\tilde{x}_k = x_k - x_{ss}$ and $\tilde{u}_k = u_k - u_{ss}$ are deviations of the plant state vector and the control input vector from the respective final steady state values of x_{ss} and u_{ss}; and $\beta(t) \in \mathbf{R}^r$ and $\Gamma \in \mathbf{R}^r$ are specified tolerances for the damage rate and accumulated damage, respectively. An open-loop control policy can be synthesized by minimizing the cost functional in equation (6) under: (i) the above inequality constraints in equation (7); and *(ii) the condition that, starting from the initial* conditions $x(t_0)$ and $v(t_0)$, the state trajectory must satisfy the plant dynamic model in equation (1). The design variables to be identified are the control inputs u_k, $k = 0, 1, 2, \ldots, N-1$, and the goal is to search for an optimal control sequence $\{u_k\}$. Development of such an open-loop control policy has been reported by Ray *et al* (1993a, b) along with the results of simulation experiments for transient operations of a reusable rocket engine.

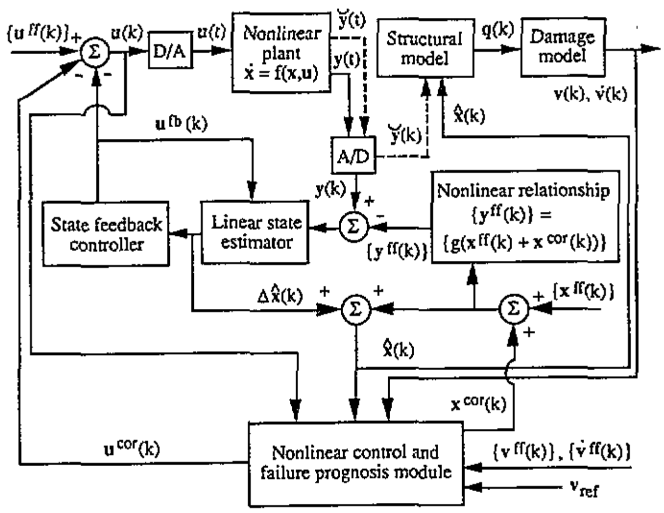

Figure 2. The closed-loop damage-mitigating control system.

2.2. The closed-loop control policy

As stated above, non-linear programming generates an *open-loop control* policy based on the nominal plant model in equation (1) to achieve optimal performance under the specified constraints of damage rate and accumulated damage. However, because of plant modeling uncertainties (including unmodeled dynamics), sensor noise and disturbances, the actual plant response shall deviate from that of the modeled system when the plant is excited by the sequence of open-loop control commands. Therefore, a closed-loop control system is necessary to compensate for these deviations, and an output feedback controller may serve this purpose. If all plant states are not measurable or if the sensor data are noise-contaminated, a state estimator (e.g., a minimum variance filter) is necessary to obtain an estimate of the plant state vector. If the deviations from the nominal trajectory are not large, the state feedback and state estimator gain matrices could be synthesized based on a linearized model of the plant. However, if the plant is required to be operated over a wide range (for example, scheduled shutdown of a power plant from full power), then linearization must be carried out at several operating points and the closed-loop control could be made piecewise linear by adopting the concept of gain scheduling. Should this concept prove to be inadequate, more advanced techniques such as (model reference or self-tuning) adaptive control (Goodwin and Sin 1984) and reconfigurable control (Stengel 1991) need to be considered.

A block diagram of the closed-loop damage-mitigating control system is proposed in figure 2 where $\{u^{ff}\}$, $\{x^{ff}\}$, $\{\upsilon^{ff}\}$ and $\{\dot{\upsilon}^{ff}\}$ represent the sequences of the plant input, plant state, accumulated damage, and damage rate, respectively, generated as an optimal solution of the open-loop control problem via non-linear programming, and $\{y^{ff}\}$ is the resulting sequence of plant output, which is obtained as a non-linear function $g(\cdot)$ of the plant state vector x^{ff}. As mentioned above, a feedback controller is necessary to compensate for the plant modeling uncertainties and disturbances such that the trajectory of the actual plant output y should be close to that of the desired plant output y^{ff} which serves as the reference trajectory. The resulting error in the plant output is an input to the state estimator, and the feedback control signal u^{fb} compensates for errors resulting from plant disturbances and sensor noise. The estimated state $\hat{x}$ which is obtained as the sum of the estimated state error and the reference state x^{ff} is fed to the structural model. The output of the structural model is the load vector q which contains stress, strain and other information necessary for damage assessment. Some of the elements of the load vector q (e.g., strain at the critical points) may be directly measurable as indicated in figure 2 by additional measurements $\tilde{y}$.

The closed-loop control system is partitioned into two modules. The state feedback control law in the first module could be formulated by using the established techniques of robust multi-input multi-output (MIMO) control synthesis, which rely on approximation of the plant dynamics by a linear time-invariant model (e.g., H_2-based LQG/LTR (Stein and Athans 1987), H_2/H_∞ optimization (Doyle *et al* 1989), or μ-synthesis (Packard *et al* 1993, Stein and Doyle 1991, Skogestad *et al* 1988). However, if the plant dynamics cannot be approximated by piecewise linearization or time-averaging of the varying parameters, then selection of the control synthesis technique will depend upon the specific application.

The second module is a non-linear controller, which contributes to damage reduction in the critical plant components under feedback control and also serves to generate early warnings and prognoses of impending failures of the critical plant components. Since the damage rate $\dot{\upsilon}$ in the closed-loop control system may violate the specified constraints due to the additional compensation generated by the feedback control effort u^{fb}, a non-linear controller is incorporated into the system to reduce the damage rate $\dot{\upsilon}$. This is accomplished by corrections x^{cor} in the reference trajectory and u^{cor} in the control effort. The rationale for modifying the reference trajectory $\{x^{ff}\}$ is that, due to plant disturbances and sensor noise, it may not be possible to follow this trajectory without violating the damage constraints. The additional control effort u^{cor} is intended to provide a fast corrective action to the control input, u, whenever necessary. The non-linear control system in the outer loop, as shown in figure 2, serves two purposes, namely, (i) trimming of the linear feedback control signal to maintain the damage rate or accumulated damage rate vector within the limits, and (ii) modification of the tracking signal (i.e., the reference signal) to circumvent the problem of exceeding the damage rate limits, which may result from plant modeling errors, uncertainties and disturbances. A possible approach to synthesis of the non-linear damage controller is first to postulate a mathematical structure for the controller and then optimize the controller parameters relative to a cost functional that would penalize the plant state and damage rate vectors over the task period along with the final plant state and the accumulated damage vectors.

The outputs of the damage controller, namely x^{cor} and y^{cor} in figure 2, must be constrained to be norm-bounded to assure the system stability; x^{cor} and y^{cor} can be considered as exogenous inputs to the linear robust control system in the inner loop. Therefore, the bounds on x^{cor} and u^{cor} can be fine-tuned to satisfy the specified requirements of performance and stability robustness in the control synthesis procedure.

3. Modeling of fatigue damage dynamics

As discussed earlier, a time-dependent model of damage dynamics, having the structure of equation (2), is necessary for analysis and synthesis of the damage-mitigating control system as shown in figure 1. From this perspective, a dynamic model of fatigue damage has been formulated in the continuous-time setting. Although this damage model has a deterministic structure, it is possible to recast it in the stochastic setting to include the effects of both unmodeled dynamics and parametric uncertainties (Sobczyk and Spencer 1992).

Because of the wide range in mechanical properties of materials, extensive varieties of experiments have been conducted for fatigue analysis, and many models have been proposed for fatigue life prediction in aircraft (Newman 1981) and ground vehicles (Tucker and Bussa 1977). Each of these models expresses the damage dynamics by an equation with the number of cycles N as the independent variable. In contrast, the damage dynamics in equation (2) are expressed as a vector differential equation with respect to time, t, as the independent variable. The advantages of this approach are that it allows the damage model to be incorporated within the constrained optimization problem and that the damage accumulated between any two instants of time can be derived even if the stress–strain hysteresis loop is not closed. This concept is applicable to different models of damage dynamics such as those resulting from cyclic strain or crack propagation. To this effect, we propose to model the continuous-time dynamics of fatigue damage based on the following two approaches.

• *Cyclic strain life.* In this approach, the local stress–strain behaviour is analysed at certain critical points where failure is likely to occur (Dowling 1983). The local strain may be directly measured from a strain gauge, or computed via finite element analysis. The local stress is estimated from the cyclic stress–strain curve. A

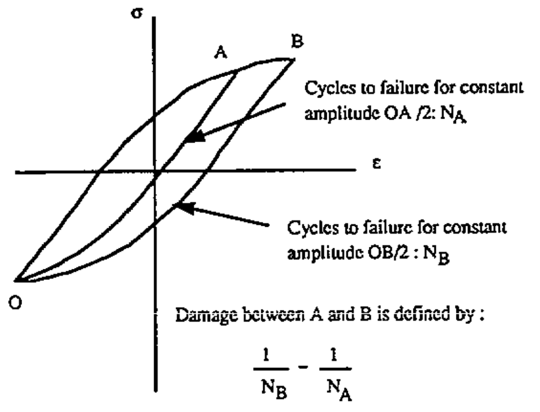

Figure 3. Damage between two points on the same reversal.

cycle-based approach is then used to estimate the fatigue damage from the strain-life curves at different levels of stress and strain in the load history. We propose to model the intra-cycle damage accumulation using the classical Palmgren–Miner rule and subsequently modify it via the damage-curve approach of Bolotin (1989).

- *Linear elastic fracture mechanics* (LEFM). The LEFM approach is built upon the concept of a physical measure of damage in terms of the crack length and the size of the plastic zone at the crack tip (Wheeler 1972, Willenborg *et al* 1971). The accumulated damage is computed by integrating the crack growth rate over the number of cycles. The component under stress is assumed to fail when the crack reaches the critical length which, in turn, is determined from the fracture toughness of the component on the basis of experimental data.

While the primary approach adopted in this paper is based on cyclic strain-life for developing a continuous-time damage model, an alternative approach based on the LEFM concept is presented in appendix B.

3.1. Damage modeling based on the cyclic strain-life concept

This section presents the concept and development of a linear model of fatigue damage accumulation in the continuous-time setting where the elastic-plastic fatigue model used in the strain-life analysis is applicable to the fatigue endurance level of approximately 10^6 cycles or less. Referring to figure 3, let the point O be the starting point of a reversal, let A and B be two consecutive points on the same rising reversal, and let N_A and N_B represent the total number of cycles to failure with constant load amplitudes, OA/2 and OB/2, respectively. Then, the half-cycle increment of the linear accumulated damage δ between points A and B is defined as:

$$\Delta\delta_{A,B} = \frac{1}{N_B} - \frac{1}{N_A}. \tag{8}$$

In equation (8), it is assumed that the damage occurs only on the rising reversal, i.e., if the stress is monotonically increasing, and no damage occurs during unloading, i.e., if the stress is monotonically decreasing. This assumption is consistent with the physical phenomena observed in the fatigue cracking propagation process. Given that $\Delta\sigma$ is the stress increment between point A and point B, the average damage rate with respect to this stress change is equal to $\Delta\delta/\Delta\sigma$. Let Δt be the time interval from A to B, the average rate of linear damage δ in terms of the stress σ can be transformed into the time domain by $\Delta\delta/\Delta t = (\Delta\delta/\Delta\sigma) \times (\Delta\sigma/\Delta t)$. Making Δt infinitesimally small, the instantaneous damage rate becomes

$$\frac{d\delta}{dt} = \lim_{\Delta t \to 0} \frac{\Delta\delta}{\Delta\sigma} \times \frac{\Delta\sigma}{\Delta t} = \frac{d\delta}{d\sigma} \times \frac{d\sigma}{dt} \tag{9}$$

where the instantaneous stress rate $d\sigma/dt$ can be generated from direct measurements of strain rate or from the finite-element analysis, and $d\delta/d\sigma$ is derived from the existing cycle-based formulae (Bannantine *et al* 1990). The strain-life and the cyclic stress–strain curves are used to evaluate $d\delta/d\sigma$ in equation (9). Replacing N_f by $1/\Delta\delta$ where $\Delta\delta$ represents the increment in linear damage δ during one cycle, the strain life relationship (Dowling 1983) in terms of the elastic damage and plastic damage modes can be written as:

$$\frac{\Delta\epsilon_e}{2} = \frac{\sigma_f' - \sigma_m}{E}\left(\frac{\Delta\delta_e}{2}\right)^{-b} \tag{10a}$$

$$\frac{\Delta\epsilon_p}{2} = \epsilon_f'\left(1 - \frac{\sigma_m}{\sigma_f'}\right)^{c/b}\left(\frac{\Delta\delta_p}{2}\right)^{-c} \tag{10b}$$

where b, c, σ_f' and ϵ_f' are material constants; σ_m is the mean stress; and the elastic and plastic strain amplitudes, $\Delta\epsilon_e/2$ and $\Delta\epsilon_p/2$, are related to the state of stress following the cyclic stress–strain characteristics:

$$\frac{\Delta\epsilon_e}{2} = \frac{|\sigma - \sigma_r|}{2E} \tag{11a}$$

$$\frac{\Delta\epsilon_p}{2} = \left(\frac{|\sigma - \sigma_r|}{2K'}\right)^{1/n'} \tag{11b}$$

where n' is the cyclic strain hardening exponent, K' is the cyclic strength coefficient and σ_r is the reference stress obtained from the cycle counting under spectral loading. From equations (10a), (10b), (11a) and (11b), the closed-form solutions for $\Delta\delta_e$ and $\Delta\delta_p$ can be obtained in terms of stress instead of strain as given below:

$$\Delta\delta_e = 2\left(\frac{|\sigma - \sigma_r|}{2(\sigma_f' - \sigma_m)}\right)^{-1/b} \tag{12a}$$

$$\Delta\delta_p = 2\left[\frac{1}{\epsilon_f'}\left(\frac{|\sigma - \sigma_r|}{2K'}\right)^{1/n'}\left(1 - \frac{\sigma_m}{\sigma_f'}\right)^{-c/b}\right]^{-1/c}. \tag{12b}$$

Step changes in the reference stress σ_r can occur only at isolated points in the load spectrum. Since the damage increment is zero at any isolated point, the damage accumulation can be evaluated at all points excluding these isolated points which constitute a set of zero Lebesgue measure (Royden 1988). Exclusion of the points of step changes in σ_r does not cause any error in the computation of damage, and $d\sigma_r/dt$ can be set to zero because σ_r is piecewise constant. Further, since it is assumed that no damage occurs during unloading, the damage rate can be made equal to zero if $\sigma < \sigma_r$. When the stress is increasing, i.e., $\sigma > \sigma_r$, the elastic damage rate $d\delta_e/dt$ and the plastic damage rate $d\delta_p/dt$ are computed from equations (12a) and (12b) in terms of the instantaneous stress rate $d\sigma/dt$ as:

if $\sigma \geqslant \sigma_r$ then

$$\frac{d\delta_e}{dt} = 2\frac{d}{d\sigma}\left[\left(\frac{\sigma - \sigma_r}{2(\sigma_f' - \sigma_m)}\right)^{-1/b}\right]\frac{d\sigma}{dt} \qquad (13a)$$

and

$$\frac{d\delta_p}{dt} = 2\frac{d}{d\sigma}\left[\left(\frac{1}{\epsilon_f'}\left(\frac{\sigma - \sigma_r}{2K'}\right)^{1/n'}\left(1 - \frac{\sigma_m}{\sigma_f'}\right)^{-c/b}\right)^{-1/c}\right] \times \frac{d\sigma}{dt} \qquad (13b)$$

else $d\delta_e/dt = 0$ *and* $d\delta_p/dt = 0$.

The damage rate $d\delta/dt$ is then obtained as the weighted average of the elastic and plastic damage rates:

$$\frac{d\delta}{dt} = w\frac{d\delta_e}{dt} + (1 - w)\frac{d\delta_p}{dt} \qquad (14)$$

where the weighing function, w, is selected on the basis of the elastic and plastic strain amplitudes in equations (11a) and (11b) as:

$$w = \frac{\Delta\epsilon_e}{\Delta\epsilon_e + \Delta\epsilon_p} \quad \text{and} \quad 1 - w = \frac{\Delta\epsilon_p}{\Delta\epsilon_e + \Delta\epsilon_p}. \qquad (15)$$

Equations (13) to (15) are then used to obtain the damage rate at any instant of time. The damage increment between two consecutive points t_k and t_{k+1} on the same reversal can be calculated by integrating the damage differential $d\delta$. The above continuous-time damage model has been verified using the experimental data generated in the SAE cumulative fatigue test program (Newman 1981) for Man–Ten steel. A comparison of the model results and experimental data is reported by Wu (1993).

The continuous-time damage model in equations (13) to (15) is derived on the basis of linear damage accumulation following the Palmgren–Miner's rule. Although this concept of linear damage accumulation has been widely used due to its simplicity in computation, the cumulative damage behaviour is actually non-linear (Bolotin 1989, Suresh 1991). Experimental results show that, for variable amplitude loading, the accumulated damage is dependent on the order in which the load cycles are applied. This phenomenon is known as the sequence effect which is further explained in the next section. Since the structural components of complex mechanical systems are subjected to loads of varying amplitude, the linear rule of damage accumulation which is commonly used for fatigue life assessment could lead to erroneous results due to this sequence effect. Therefore, a non-linear damage rule needs to be established for accurate prediction of the damage rate and damage accumulation in the critical components.

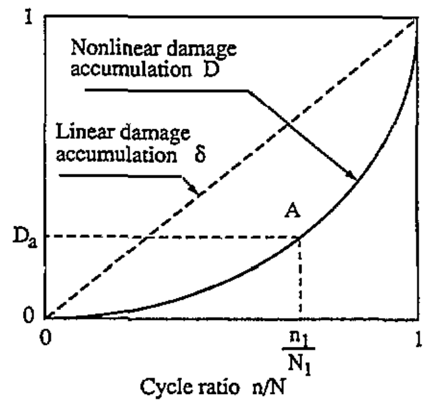

Figure 4. Nonlinear damage curve under constant amplitude loading.

3.2. Modeling of non-linear cumulative damage using the damage-curve approach

The concept of a non-linear damage curve to represent the damage was first conceived by Marco and Starkey (1954). No mathematical representation of the damage curve was proposed at that time because the physical process of damage accumulation was not adequately understood. Manson and Halford (1981) proposed the double linear rule primarily based on the damage-curve approach for treating cumulative fatigue damage. In their paper, an effort was made to mathematically represent the damage curve and approximate it by two piecewise line segments. The total fatigue life was then divided into two phases so that the linear damage rule can be applied in each phase of the life. A concept similar to the damage curve approach was proposed by Bolotin (1989) with a mathematical representation which does not necessarily assess the damage on the basis of cycles and is more appropriate for modeling in the continuous-time setting. Therefore, we have adopted Bolotin's approach in this research to develop a continuous-time model of non-linear damage accumulation.

Figure 4 shows a comparison of the accumulations of linear damage, δ, and non-linear damage, D, as a function of the cycle ratio, n/N, where n is the actual number of cycles undergone and N is the number of cycles to failure under a constant amplitude loading. The damage accumulates along the curve as the loading cycles are applied. For example, if n_1 cycles of a constant stress amplitude, which has a fatigue life of N_1, are applied to the component, the accumulated damage will follow the curve and reach D_a as indicated in figure 4 where the abscissa is the normalized cycle ratio with respect to its fatigue life and the ordinate is the physical damage accumulation of the component. Bolotin (1989) used the following analytical relationship between D and δ

$$D = (\delta)^{\gamma} \qquad (16)$$

where the component γ describes the non-linearity of the curve and is usually a function of the stress amplitude σ_a. The rationale for distinct γ parameters under different loading conditions can be justified as follows.

Let a smooth specimen be subjected to two levels of stress amplitude, $\Delta\sigma_1$ and $\Delta\sigma_2$, with respective fatigue lives of N_1 and N_2 cycles. Let $k_1 = 1/N_1$ and $k_2 = 1/N_2$ be the linear damage increments per cycle for $\Delta\sigma_1$ and $\Delta\sigma_2$ respectively. If γ is assumed to be a constant, then the damage rate with respect to the number of cycles is obtained by differentiation of equation (16) as:

$$\frac{\mathrm{d}D}{\mathrm{d}N} = \gamma\delta^{\gamma-1}\frac{\mathrm{d}\delta}{\mathrm{d}N}. \tag{17}$$

If n_1 cycles of $\Delta\sigma_1$ are applied on the specimen and followed by $\Delta\sigma_2$ for n_2 cycles, then the total damage accumulation can be computed as

$$\begin{aligned} D &= \int_0^{n_1} \gamma\left(k_1 n\right)^{\gamma-1} k_1 \mathrm{d}n \\ &\quad + \int_0^{n_2} \gamma\left(k_1 n_1 + k_2 n\right)^{\gamma-1} k_2 \mathrm{d}n \\ &= k_1^\gamma n_1^\gamma \left[\left(k_1 n_1 + k_2 n_2\right)^\gamma - \left(k_1 n_1\right)^\gamma\right] \\ &= \left(k_1 n_1 + k_2 n_2\right)^\gamma. \end{aligned} \tag{18}$$

Next, let n_2 cycles of $\Delta\sigma_2$ be followed by n_1 cycles of $\Delta\sigma_1$. Then, the total damage accumulation is

$$\begin{aligned} D &= \int_0^{n_2} \gamma\left(k_2 n\right)^{\gamma-1} k_2 \mathrm{d}n \\ &\quad + \int_0^{n_1} \gamma\left(k_2 n_2 + k_1 n\right)^{\gamma-1} k_1 \mathrm{d}n \\ &= k_2^\gamma n_2^\gamma + \left[\left(k_1 n_1 + k_2 n_2\right)^\gamma - \left(k_2 n_2\right)^\gamma\right] \\ &= \left(k_1 n_1 + k_2 n_2\right)^\gamma. \end{aligned} \tag{19}$$

As seen from equations (18) and (19), there is no change in damage accumulation if γ is a constant. However, this is inconsistent with the phenomenon observed from the experiments, which show significant differences in accumulated damage due to sequence effect. Hence, the γ-parameter cannot be assumed to be a constant as it must have distinct values for different load conditions. Consequently, the non-linear damage accumulation follows distinct curves under different stress amplitudes. For multiple-level loading, the accumulated damage is computed by first identifying the current damage state and then following the damage curve associated with the current loading condition. If a component is subjected to spectral loading, then the technique of cycle-counting, prediction of linear damage increments, and computation of non-linear damage via the damage curve approach need to be integrated into a single procedure.

In the damage-curve approach above, the γ-parameter is often assumed to be dependent on the stress amplitude level (Manson and Halford 1981). High-strength materials such as 4340 steel usually yield large values of γ especially under high-cycle fatigue. From equation (16), it follows that a large γ initially creates a small damage, which could be out of the range of precision of the computer. This indicates that the accumulated damage behaviour may not be accurately described by the non-linear damage curve with γ-parameter solely dependent on the stress amplitudes, especially at the early stage of the fatigue life. It follows from a crack propagation model such as the Paris model (Paris and Erdogan 1963) that the crack growth rate is dependent not only on the stress amplitude but also on the current crack length as explained below.

Recognizing the fact that the crack itself is often viewed as an index of accumulated damage, the accumulated damage can be defined as $D = a/a^*$, where a^* is the critical crack length for the component to fail. Then, following the LEFM approach for continuous-time damage modeling delineated in appendix A, the damage rate based on the Paris equation is given as

$$\frac{\mathrm{d}a}{\mathrm{d}t} \sim \left(\sigma_a\sqrt{a}\right)^n \Rightarrow \frac{\mathrm{d}D}{\mathrm{d}t} \sim \sigma_a^n D^{n/2}. \tag{20}$$

For a constant stress amplitude σ_a, the linear damage δ is proportional to the time t. Therefore, it follows from equation (16) that the non-linear accumulated damage $D(t) \sim t^\gamma$ with γ being a constant. Then, setting $D(t) = Ct^\gamma$ where C is the constant of proportionality, the damage rate becomes

$$\begin{aligned} &\frac{\mathrm{d}D}{\mathrm{d}t} = \gamma C t^{\gamma-1} = \gamma C^{1/\gamma}\left(Ct^\gamma\right)^{(\gamma-1)/\gamma} \\ &\quad\Rightarrow \frac{\mathrm{d}D}{\mathrm{d}t} = \gamma C^{1/\gamma}(D)^{(\gamma-1)/\gamma}. \end{aligned} \tag{21}$$

For equations (20) and (21) to be identical for all D, it is necessary that $n/2 = (\gamma-1)/\gamma$. However, since n is usually greater than 2 (see Fuchs and Stephens 1980) and $(\gamma-1)/\gamma < 1$ because $\gamma > 0$, it may not be possible to match equation (21) with the continuous-time crack growth equation (20) for most of the materials. Hence, γ should not be independent of the current level of damage accumulation implying that $\gamma = \gamma(\sigma_a, D)$. Accordingly, a modification of equation (16) is proposed as follows:

$$D = (\delta)^{\gamma(\sigma_a, D)} \tag{22}$$

where D and δ are the current states of non-linear and linear damage accumulation respectively. Although the above equation (22) has an implicit structure, it can be solved via a recursive relationship.

The next part of this section describes a modification of the damage-curve approach to develop a non-linear damage model in the continuous-time setting. Following the concept of the linear damage model in the continuous-time setting, the non-linear damage at any point on a rising reversal can be obtained as explained below.

Referring to the bottom part of figure 5, let A be any point on the rising reversal and R be its reference point as determined from the rainflow cycle counting method (Dowling 1983, Rychlik 1993). Let the current state of damage at the reference point R be equal to D_r and let ORAZ be the damage curve associated with the stress amplitude RA/2 as shown in the top part of figure 5.

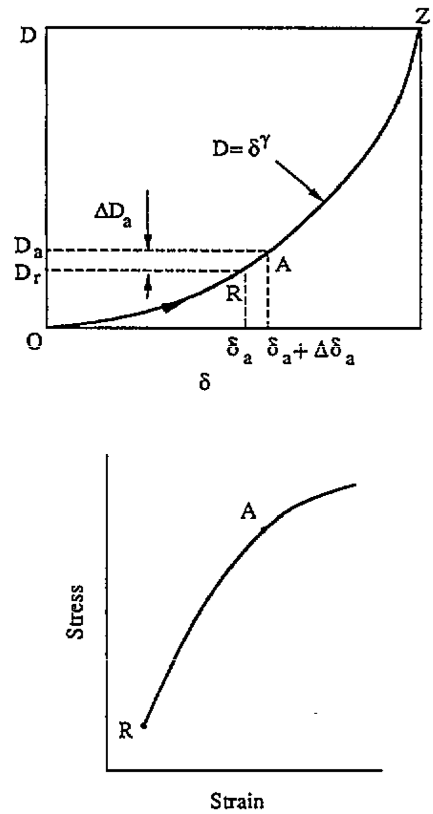

Figure 5. Nonlinear damage increments.

Corresponding to the non-linear damage D_r, we bring in the notion of the 'virtual' linear damage, δ_a, at the point R as follows:

$$\delta_a = \left(D_r\right)^{1/\gamma_r^a} \tag{23}$$

where γ_r^a is the γ-parameter associated with the stress amplitude RA/2 and non-linear accumulated damage D_r at R. The term 'virtual' means that δ_a is the linear damage which would be incurred if the component had been subjected to the cyclic stress of constant amplitude, RA/2, from its initial damage state to the current damage state at R. Similar to γ_r^a in equation (23), we define γ_r^a as the γ-parameter associated with the damage state D_a at point A.

Referring to figure 5, the functional relationships among D_a, D_r and the corresponding γ-parameters γ_a^a and γ_r^a are defined as follows:

$$\begin{cases} \gamma_a^a = \gamma(\sigma_a, D_a) \\ D_a = (\delta_a + \Delta\delta_a)^{\gamma_a^a} \end{cases} \quad \text{and} \quad \begin{cases} \gamma_r^a = \gamma(\sigma_a, D_r) \\ D_r = (\delta_a)^{\gamma_r^a} \end{cases} \tag{24}$$

where σ_a is the stress amplitude RA/2, and $\Delta\delta$ is the linear damage increment between R and A obtained via the procedure described in section 3.1. The damage D_r in the right-hand part of equation (24) represents the damage state at the reference point R and therefore its value is already computed from the past load history. Knowing D_r, the γ-parameter γ_r^a and the 'virtual' linear damage δ_a at R for the stress amplitude RA/2 can be evaluated from the right-hand part of equation (24). The two unknowns, D_a and γ_a^a, which represent the damage state at the current point A, are computed by solving the equation pair in the left-hand part of equation (24), and the issue of existence and uniqueness of the solution is discussed later in this section. Now, the (non-linear) damage increment from the point R to A in figure 5 can be computed as:

$$\Delta D_a = D_a - D_r = \left(\delta_a + \Delta\delta_a\right)^{\gamma_a^a} - \delta_a^{\gamma_r^a}. \tag{25}$$

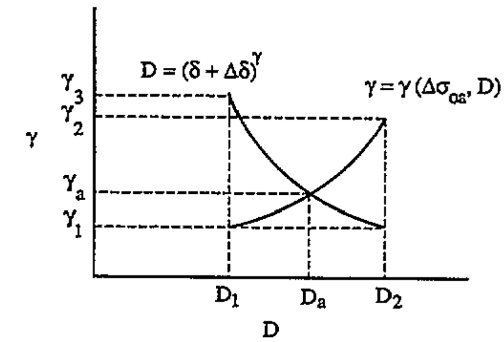

Figure 6. Computation of non-linear cumulative damage.

In summary, the accumulated damage at any point within a reversal can be obtained by solving the following non-linear equations:

$$\gamma = \gamma\left(\sigma_a, D\right) \tag{26}$$

$$D = (\delta + \Delta\delta)^{\gamma} \tag{27}$$

where δ is the 'virtual' linear damage at the reference point, and $\Delta\delta$ is the linear damage increment for the stress amplitude σ_a relative to the reference point.

The pair of equations (26) and (27) does not have a closed form solution and therefore needs to be solved by an iterative method. However, an iterative solution at every point in the load history may not be practical from the perspective of computational efficiency. An efficient approach of obtaining a numerical solution would operate on a set of discrete points in the local history such that there is only a very small increment of damage between two consecutive points. Thus, equations (26) and (27) can be treated as linear line segments between the points R and A. This assumption is valid during the entire loading history with the possible exception of very low-cycle fatigue. One more interesting observation is that the γ-parameter is generally a monotonically increasing function of the non-linear damage D. This can be interpreted from the Paris equation whereby the growth rate of a macrocrack becomes larger as the crack length increases. Therefore, the damage rate would be larger at a higher degree of non-linearity, which implies a larger value of γ. This phenomenon, however, may not be true if the stress intensity factor range is below the long-crack threshold or if the material strain hardens. For the general case of $\gamma > 1$ (e.g., high strength materials that usually strain soften), γ is a monotonically increasing function of D in equation (26), and D is a monotonically decreasing function of γ in equation (27). As seen in figure 6, this equation set has a unique solution at D_a where $D_1 < D_a < D_2$ and $D_2 = (\delta+\Delta\delta)^{\gamma_1}$. On the other hand, if γ is less than 1 (e.g., ductile materials that usually strain harden), the characteristics of both equations (26) and (27) could be reversed.

Having computed the linear damage δ and linear damage increment $\Delta\delta$, a procedure for solving equations (26) and (27) to obtain the non-linear damage, D, is described below:

1. Let $D_1 = D_r$ and $\gamma_1 = \gamma_r$.

2. Compare $D_2 = (\delta + \Delta\delta)^{\gamma_1}$ from equation (27) and $\gamma_2 = \gamma(\sigma_a, D_2)$ from equation (26).

3. Compute $\gamma_3 = \gamma_1 \times [\log\delta/\log(\delta + \Delta\delta)]$ from equations (22) and (26).

4. Find D_a in figure 6, which is approximated as the point of intersection of two straight line segments:

$$D_a = \frac{D_1(\gamma_2 - \gamma_1) + D_2(\gamma_3 - \gamma_1)}{\gamma_3 + \gamma_2 - 2\gamma_1}. \qquad (28)$$

The above procedure computes the non-linear accumulated damage at any point on the rising reversal. If the stress is monotonically decreasing, there is no damage increment as described in equations (13a) to (13b) for the case of linear damage accumulation. Finally, the rate of non-linear damage accumulation is obtained directly by differentiating equation (22) with respect to time t:

$$\frac{dD}{dt} = \gamma(\delta)^{\gamma-1}\frac{d\delta}{dt} + (\delta)^{\gamma}\ln\delta\frac{d\gamma}{dt}. \qquad (29)$$

3.3. The γ-parameter fitting for the non-linear cumulative damage model

One major task in the above approach is to identify a mathematical representation for the γ-parameter as a function of the stress amplitude and the current damage state. It requires knowledge of the physical process of damage accumulation which may be obtained from either experimental data or a combination of experimental data and analysis, and an appropriate definition of damage. The γ-parameters are different, in general, for different materials and follow different structures of the governing equations. Furthermore, because the mechanisms attributed to the damage accumulation at various stages of fatigue life are different, no single approach apparently provides a sufficiently accurate prediction of damage throughout the fatigue life of a component. It is difficult, if not impossible to construct a single structure for representation of γ. An alternative approach is to evaluate γ by interpolation. Once the damage is appropriately defined, and an analytical method for damage prediction is selected, the information needed for the non-linear damage model can be generated via experimentation or analysis for a set of constant stress amplitudes. The values of γ are then computed by equation (22) for a given stress amplitude and the damage data ranging from the initial damage state D_0 to the failure condition at $D = 1$. The generated data, γ versus D, are then plotted for various amplitudes of stresses. These curves can be either fitted by non-linear equations, if possible, or as linear piecewise representations. Since it is not practical to perform experiments at infinitesimally small increments of stress amplitude, γ can only be experimentally evaluated at selected discrete levels of stress amplitude. The values of γ for other stress amplitudes can then be interpolated. Since the characteristics of γ may strongly depend on the type of the material, availability of pertinent experimental data for the correct material is essential for the damage-mitigating control synthesis. The remaining part of this section presents the results of γ-parameter fitting based on the experimental data (Swain *et al* 1990) for the material AISI 4340 steel.

Table 1. Effective stress intensity factor range versus crack growth rate relationship.

ΔK_{eff} (MPa $m^{1/2}$)	da/dN (m/cycle)
3.75	3.0E-10
5.30	2.0E-09
7.30	7.0E-09
15.00	4.5E-08
50.00	5.5E-07
120.00	3.0E-05

In the model of Newman *et al* (1992), the stress intensity factor range ΔK used in the Paris equation is replaced by the effective stress intensity factor range ΔK_{eff}. The crack opening stress is determined by a crack-closure model which is similar to the Dugdale model (Dugdale 1960) but it is modified to leave plastically deformed material in the wake of the advancing crack tip. In the research reported in this paper, however, we have computed this crack opening stress by simplified equations (Newman 1984) which are obtained through curve fitting based on the original model. The experiments show that a unified approach (Newman *et al* 1992) based on the crack-closure concept can be used for damage prediction starting from the initial defect (microcrack) to the failure of materials without significant errors. A relationship between the effective stress intensity factor range and crack growth rate obtained from the experimental data of AISI 4340 steel are given in table 1 following Swain *et al* (1990).

In table 1, the values of ΔK_{eff} and da/dN are linearly interpolated between two consecutive data points using the logarithmic scale while those beyond the maximum and minimum are extrapolated. The initial crack size is identified based on the assumption that the fatigue life predicted by the strain-life approach is equal to that obtained via the crack closure model. The damage, in this case, is defined as the normalized crack length with respect to its critical length to failure, i.e., $D = a/a^*$ where the critical crack length a^* is obtained from fracture toughness of the material. The material and fatigue properties of AISI 4340 steel are given in table 2 following Boller and Seeger (1987).

The results of the non-linear damage D versus the linear damage δ which is essentially the cycle ratio are used to compute γ from equation (22). Figure 7 shows the relationship between γ and D for different

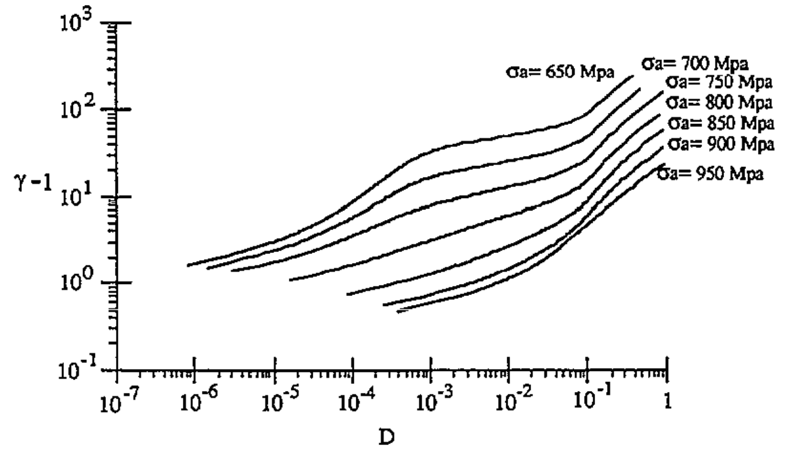

Figure 7. γ versus D at different stress amplitudes.

Table 2. Material properties of AISI 4340 steel.

Young's modulus (E)	193500 N mm^{-2}
Yield strength, monotonic (σ_y)	1374 N mm^{-2}
Yield strength, cyclic (σ'_y)	905 N mm^{-2}
Cyclic-strength coefficient (K')	1890 N mm^{-2}
Cyclic-strength hardening exponent (n')	0.118
Fatigue-strength coefficient (σ'_f)	1880 N mm^{-2}
Fatigue-strength exponent (b)	−0.086
Fatigue-ductility coefficient (ϵ'_f)	0.706
Fatigue-ductility exponent (c)	−0.662

values of the (constant) stress amplitude σ_a as a series of curves in the logarithmic scale. If these curves are generated to be closely spaced, the values between two curves can be obtained via linear interpolation without an appreciable error. Once the γ-parameters are computed as a function of two independent variables $\Delta\sigma$ and D, the non-linear damage model described above can be readily used to predict the non-linear damage behaviour at different stress amplitudes.

4. Summary and conclusions

The motivation of the work reported in this paper was to achieve high performance in complex processes (e.g., advanced aircraft, spacecraft, power plants, and chemical plants) without overstraining the mechanical structures such that the functional life of critical components is increased resulting in enhanced safety, operational reliability, and availability. In the proposed approach, the damage rate and accumulated damage need to be constrained for optimization of performance and structural durability as reported by Ray *et al* (1993b) and Wu (1993). This requires a fatigue damage model which is comparable with the state-space representation of the controlled process. Therefore, the damage model is described by differential equations with time as the independent variable instead of the conventional cycle-based approach. Since the parameters of the fatigue damage model strongly depend on the type of material, availability of pertinent experimental data for the correct material is essential for service life prediction, risk analysis, and control synthesis.

The proposed approach to damage-mitigation in mechanical structures has a wide spectrum of potential engineering applications. Examples are reusable rocket engines for space propulsion, rotating and fixed wing aircraft, fossil and nuclear plants for electric power generation, large chemical and processing plants, automotive and truck engine/transmission systems, and large rolling mills. In each of these systems, damage-mitigation can enhance service life, safety and productivity. Therefore, extended life coupled with enhanced safety and high performance is expected to have a significant economic impact in diverse industrial applications.

Acknowledgments

The authors acknowledge benefits of discussion with Carl F Lorenzo of NASA Lewis Research Center, and Professors Marc Carpino and D A Koss of Penn State University.

The research work reported in this paper is supported in part by NASA Lewis Research Center under Grant No NAG 3-1240, National Science Foundation under Grant No ECS-9216386, National Science Foundation under Research Equipment Grant No MSS-9112609, Office of Naval Research under Grant No N00014-90-J-1513, and Electric Power Research Institute under Contract No EPRI RP8030-5.

Appendix A. Failure prognosis and risk analysis

Failure prognosis is envisioned to be an augmentation of the failure diagnosis scheme within the domain of *Intelligent Control Systems* (Stengel 1991) to predict and locate impending failures of structural components. In addition to the modeling errors of damage dynamics, prediction of both the damage rate $\dot{\nu}$ and hence the accumulated damage v are affected by plant disturbances and sensor noise. The damage calculations are based on the estimated state $\hat{x}$ unless the additional measurements $\tilde{y}$, such as the local strain and temperature, are available. Therefore robustness of the prognostic decisions is dependent on uncertainties in each of the plant, structural and damage models as seen in figure 2, and these decisions should be made on the basis of current and past observations of v, $\hat{x}$ and u. In essence, the failure prognosis system must be cognizant of the plant operational requirements, mission objectives, and the limitations and inaccuracies of the predicted damage information.

Traditionally, the risk index and residual service life (Bolotin 1989) of mechanical structures and machine components are calculated off-line on the basis of statistical models of the mechanical properties of the material, operating conditions, disruptions, and accidents (e.g., seismic and large atmospheric disturbances). It follows from equation (2) that the service life is finite as the damage accumulation is monotonic with time. Furthermore, the plant operations and maintenance are scheduled on the basis of the fact that the damage accumulation is likely to be accelerated in the event of accidents or disruptions. Therefore, on-line monitoring of the accumulated damage will allow refinements of the risk index and residual service life estimates (Bogdonoff and Kozin 1985, Sobczyk and Spencer 1992) as time progresses. Specifically, predictions of the accumulated damage and the current damage rate will assist the plant engineer in making dynamic decisions regarding the safety procedures, mission accomplishments, and the time interval between major maintenance actions. The concept of service life evaluation via on-line monitoring is outlined below.

Let $\varphi_j(t)$, $j = 0, 1, 2, \ldots, m$ represent predicted occurrences of impending accidents or major disruptions after the current instant of time, t, such that the set φ_0 indicates the absence of any accidents or disruptions, and the event sets φ_j, $j = 0, 1, 2, \ldots, m$ are exhaustive and mutually exclusive. Then, in the absence of any accidents or disruptions, the service life θ_0 at time t is obtained in the deterministic setting in terms of the plant state estimate $\hat{x}(t)$ and damage state $v(t)$ as:

$$\theta_0(t, \hat{x}(t), v(t)) = \mathrm{Sup}\{\tau \in [t, \infty) \text{ such that } v(\tau) \in \Omega \text{ conditioned on occurrence of } \varphi_0(t)\} \quad \text{(A1)}$$

where Ω is the set of the admissible damage vectors for the plant to remain in service. Obviously, the service life may be reduced if an accident or disruption occurs. That is, $\theta_j \leqslant \theta_0$ for $j = 1, 2, \ldots, m$. Hence, the service life θ_j based on the occurrence of the event $\varphi_j(t)$ is obtained in the probabilistic setting as:

$$\theta_j(t, \hat{x}(t), v(t)) = \sup\{\tau \in [t, \theta_0] \text{ such that } \Pr\{v(\tau) \in \Omega | \varphi_j(t)\} \geqslant (1 - P)\}, \quad j = 1, 2, \ldots, m \quad \text{(A2)}$$

where $0 < P < 1$ is the threshold probability that the damage vector may violate the admissible set Ω during the service period, i.e., while the plant is in operation. Then, the expected service life at time, t, can be obtained as:

$$\Theta(t, \hat{x}(t), v(t)) = \sum_{j=0}^{m} \theta_j(t, \hat{x}(t), v(t))\pi_j(t) \quad \text{(A3)}$$

where $\pi_j(t)$, $j = 0, 1, 2, \ldots, m$ are the *a priori* probabilities of occurrence of the events φ_j over the interval $[t, \theta_0]$.

In the absence of any accident or disruption during the period remaining after the current time t, the service life θ_0 in equation (A1) can be obtained by solving equations (1) and (2) in section 2. However, computation of θ_j for $j = 1, 2, \ldots, m$ in equation (A2) is not straightforward because the damage accumulation depends not only on the variations in the plant and damage dynamics as a result of the accident or disruption but also on the instant(s) of its occurrence. To predict the service life from equation (A2), probabilistic risk analysis is necessary beyond those reported in the literature (Bolotin 1989, Bogdonoff and Kozin 1985, Sobczyk and Spencer 1992). This would require non-linear stochastic modelling of the plant and damage dynamics. Since analytical solutions of the non-linear stochastic differential equations excited by randomly occurring discrete events may not be achievable, the idea is to generate approximate solutions from simplified models supported by the available sensor data history. These results, in turn, are to be verified by simulation.

The major implications of the proposed service life prediction and risk analysis are: (i) generation of early warnings resulting in diminished risk of loss of human life and plant equipment; (ii) enhancement of plant availability by avoiding an emergency shutdown; and (iii) reduction of the plant operational cost by dynamic rescheduling of the maintenance plan via on-line assessment of damage in the critical components.

Appendix B. Modeling of fatigue damage via a linear elastic fracture mechanics approach

The concept of a continuous-time damage model, developed in section 3 using the strain-life approach, can be extended to formulate a continuous-time model for crack propagation as explained below.

In the most commonly used fatigue crack model, e.g., the Paris model (see Bannantine *et al* 1990), the growth rate of the crack length, a, relative to the number of cycles, N, is directly dependent on the stress intensity factor range ΔK. We propose following the concept of the crack-closure model (Newman 1981) and therefore ΔK is replaced by the effective stress intensity range ΔK_{eff}. To be consistent with the strain-life approach, damage is assumed to be identically equal to l at the instant of component failure, i.e., when the crack length, a, reaches the critical value, a^*, which can be estimated from the fracture toughness of the material. The damage is assumed to occur only if the instantaneous stress is increasing and the crack opening stress σ_0 is exceeded.

This implies that the damage rate is equal to zero during unloading. The damage increment, denoted as ΔD_f, during the rising reversal can be expressed as a function of σ, σ_0 and D_f

$$\Delta D_f = \frac{\mathrm{d}D_f}{\mathrm{d}N} = \frac{1}{a^*}\frac{\mathrm{d}a}{\mathrm{d}N} = \frac{1}{a^*}F\big(\sigma, \sigma_0, a^* D_f\big) \qquad \text{(B1)}$$

where the fatigue damage is defined as $D_f = a/a^*$ in the absence of a more precise definition of damage. Since the damage increment ΔD_f is relatively small, the total damage D_f is assumed to remain constant during a reversal. The damage rate can be obtained by differentiating equation (A1) with respect to time.

$$\frac{\mathrm{d}D_f}{\mathrm{d}t} = \frac{1}{a^*}\frac{\partial F(\sigma, \sigma_0, a^* D_f)}{\partial \sigma}\frac{\mathrm{d}\sigma}{\mathrm{d}t} \qquad \text{(B2)}$$

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