

# A nonlinear stochastic model of fatigue crack dynamics

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This paper presents a nonlinear stochastic model for prediction of fatigue crack damage in metallic materials. The model structure allows estimation of the current damage state and prediction of the remaining service life based on the underlying principle of Gauss–Markov processes without solving the extended Kalman filter equation in the Wiener integral setting or the Kolmogorov forward equation in the Itô integral setting. The model results have been verified with experimentally-generated statistical data of time-dependent fatigue cracks for 2024-T3 and 7075-T6 aluminum alloys. Copyright © 1996 Elsevier Science Ltd.

## 1 INTRODUCTION

Dynamic modeling of fatigue crack damage has been a topic of intensive research for several decades. Many researchers, cited by Suresh,<sup>1</sup> have proposed empirical and semi-empirical models in the deterministic setting based on observed experimental data and attempted to provide a physical interpretation to these models. However, the sole usage of deterministic methods that model the mean path of the fatigue crack growth process fails to adequately represent the variability in estimated damage and predicted service life. This is evident even in experiments conducted in a controlled laboratory environment.<sup>2,3</sup>

Modeling of fatigue crack growth via nonlinear stochastic differential equations is a relatively new area of research, and a list of the literature representing the state of the art is cited in Sobczyk and Spencer.<sup>4</sup> One approach to stochastic modeling of fatigue crack growth is to randomize the coefficients of an established deterministic model to represent the material inhomogeneity.<sup>5,6</sup> Another approach is to generate the necessary stochastic information by multiplying the deterministic dynamics of fatigue crack growth with a nonnegative random process.<sup>7–12</sup> The process of fatigue crack growth is thus modeled by nonlinear stochastic differential equations in the Itô setting.<sup>13</sup> Specifically, Kolmogorov forward and backward diffusion equations, which require solutions of nonlinear partial differential equations, have been proposed to generate the statistical information required for risk analysis of mechanical

structures.<sup>14,15</sup> These nonlinear partial differential equations can only be solved numerically; the numerical procedures, however, are computationally intensive as they rely on a fine-mesh model using finite-element or combined finite-difference and finite-element methods.<sup>16</sup> Therefore, although this numerical approach might be useful for making off-line decisions for design analysis and predictive maintenance, it is not sufficiently fast for on-line damage monitoring, failure prognosis, and remaining service life prediction. To enhance the computational efficiency for on-line execution of the damage estimation and life prediction algorithms, Ray and Tangirala<sup>17</sup> have proposed an algorithm for real-time estimation of crack damage based on the underlying principle of extended Kalman filtering.<sup>18</sup> In this approach, the first two moments of the stochastic damage state are computed on-line by constructing the stochastic differential equations in the Wiener form as opposed to the Itô form. The damage estimation algorithm follows the two-state model structure,<sup>9</sup> where the shaping filter is constructed with additive white Gaussian noise. The concept of extended Kalman filtering can be used with or without any sensor(s) for crack length measurement. The absence of any useful sensor data is equivalent to having the inverse of the intensity of measurement noise covariance tend to zero; in that case, the filter gain approaches zero. Consequently, the conditional density function generated by the filter becomes identical to the prior density function whose evolution is governed by the Kolmogorov forward equation.<sup>18</sup>

This paper proposes an alternative concept for stochastic modeling of fatigue crack damage, based on the underlying principle of Gauss–Markov processes, that does not require solutions of the extended Kalman filter equation in the Wiener integral setting or the Kolmogorov forward equation in the Itô integral setting. The model predictions are shown to be in close agreement with the results generated from experimental data of fatigue tests<sup>2,3</sup> for 2024-T3 and 7075-T6 aluminum alloys. The proposed model is significantly computationally faster than other reported stochastic models including our earlier model<sup>17,19</sup> and is suitable for on-line monitoring of fatigue crack damage in metallic materials which are commonly encountered in mechanical, power and chemical plants. The model verification so far has been restricted to constant amplitude cyclic loading at normal ambient conditions due to limited availability of random fatigue test data. It is recognized that the physics of fatigue damage under constant amplitude cyclic loading at normal ambient conditions is, in most cases, significantly different from that under varying amplitude loading at elevated temperatures and corrosive environment prevalent in actual plants. Nevertheless, the research work reported in this paper is a crucial step toward achieving the ultimate goal of constructing on-line damage monitoring systems which will be functional in the actual environments of plant operations. The objective here is to obtain a clear understanding of the random phenomena in fatigue and fracture of metallic materials and to establish a framework for on-line damage monitoring in the continuous-time setting.

The paper is organized in four sections including the introduction. Section 2 presents dynamic modeling of fatigue cracks in the stochastic setting. Section 3 verifies the model results with experimental data. Finally, the paper is summarized and concluded in Section 4.

## 2 DYNAMIC MODELING OF FATIGUE CRACK GROWTH

The deterministic fatigue crack growth model, adopted in this paper, is based on the short crack phenomena.<sup>20,21</sup> While the Paris model<sup>22</sup> is valid in the macro-crack range, the Newman model represents the fatigue crack growth process down to micro-cracks of the order of material defect sizes. The Newman model for crack growth is of the following form:

$$\frac{d\mu_c}{dN} = C_1(\Delta K_{\text{eff}})C_2 \left( \frac{1 - \left(\frac{\Delta K_0}{\Delta K_{\text{eff}}}\right)^2}{1 - \left(\frac{K_{\text{max}}}{C_5}\right)^2} \right) \quad (1)$$

for  $N \geq N_0$  and  $\mu_c(N_0) > 0$

where  $\mu_c$  is the mean crack length;  $N$  is the number

of cycles;  $d\mu_c/dN$  is the so-called derivative of  $\mu_c$  with respect to  $N$  as commonly used in the fracture mechanics literature;<sup>1</sup> and  $\Delta K_0 = C_3(1 - C_4 \frac{S_o}{S_{\text{max}}})$ ,  $K_{\text{max}} = S_{\text{max}} \sqrt{\pi \mu_c} F$ ,  $\Delta K_{\text{eff}} = (S_{\text{max}} - S_o) \sqrt{\pi \mu_c} F$ ;  $S_o$  is the crack opening stress;  $F$  is the correction factor for finite width of the specimen; and  $S_{\text{max}}$  is the maximum applied remote stress. Alternatively, the crack growth rate can be determined by the look-up table as a function of  $\Delta K_{\text{eff}}$ . In this setting, the crack growth rate is expressed as

$$\frac{d\mu_c}{dN} = \exp(m \ln(\Delta K_{\text{eff}}) + b) \quad (2)$$

for  $N \geq N_0$  and  $\mu_c(N_0) > 0$

where  $m$  is the slope and  $b$  is the intercept of the linear interpolation of the (log scale)  $\Delta K_{\text{eff}}\text{-dc}/dN$  look-up table. Details of this method are reported by Newman *et al.*<sup>21</sup> It should be noted, however, that any deterministic fatigue crack growth law can be used in this formulation with the only criterion being that the observed experimental crack growth profile is accurately represented. For example, several researchers<sup>9,10,16</sup> have used a cubic polynomial fit in  $\ln(\Delta K_{\text{eff}})$  to determine crack growth rate. Following Ray *et al.*,<sup>23</sup> the crack growth equation is expressed in the continuous-time setting as

$$\frac{d\mu_c}{dt} = \frac{(\partial \Phi / \partial S_{\text{max}})(dS_{\text{max}}/dt)}{1 - \partial \Phi / \partial \mu} \quad (3)$$

for  $t \geq t_0$  and  $\mu_c(t_0) > 0$

where eqn (1) or (2) is represented as  $d\mu_c/dN = \Phi(S_{\text{max}}, S_o, \mu_c)$ . The time-dependent fatigue damage may then be obtained by normalizing the crack length with respect to the critical crack length which is a function of the material, component geometry, and the operating condition.

Most of the continuous-time diffusion process models of stochastic fatigue crack growth rate reported in the literature assume that the crack growth rate is a lognormal distributed stochastic process. This is a direct consequence of the fact that most deterministic fatigue crack growth models compute the crack growth rate as a function of the applied stress, current crack length, ambient conditions, and other factors.<sup>4,7</sup> Furthermore, this assumption is motivated by the physical reality that crack growth rate can never be negative, i.e. crack healing is impossible unless the damaged component is repaired or replaced. Traditionally, the stochastic differential equation for the crack growth process  $dc(\omega, t)/dt$  is expressed in terms of the deterministic damage dynamics as

$$\frac{dc(\omega, t)}{dt} = \exp\left(x(\omega, t) - \frac{\sigma_x^2(t)}{2}\right) \frac{d\mu_c}{dt} \quad \forall t \geq t_0$$

given  $E[c(\omega, t_0)] = \mu_c(t_0)$  and  $\text{Cov}[c(\omega, t_0)] = P_0$  (4)

$$\frac{dx(\omega, t)}{dt} = -\xi x(\omega, t) + w(\omega, t) \quad \forall t \geq t_0$$

given  $E[x(\omega, t_0)] = x_0$  and  $\text{Cov}[x(\omega, t_0)] = \frac{Q_0}{2\xi}$  (5)

where  $\omega$  and  $t$  represent the sample point and time of the stochastic process, respectively; the filter parameter  $\xi$  is a measure of ‘coloredness’ of the auxiliary random process  $x(\omega, t)$ ; the correlated stationary Gauss–Markov process  $x(\omega, t)$  is modeled by using a first-order linear shaping filter which is driven by additive zero-mean white Gaussian noise  $w(\omega, t)$  of intensity  $Q_0$ . This implies that the crack growth rate is a lognormal distributed Markov process.

There are certain shortcomings of the lognormal distributed crack growth (LDCG) model described by (4) and (5). It is very difficult, if not impossible, to establish validity of the assumption of lognormal distribution due to the noise capture phenomenon which may occur whenever a numerical difference method is applied to obtain crack growth rates from experimental data. Moreover, there is a finite probability that, for a given sample, the fatigue crack growth rate may be large enough to completely rupture the structure in one cycle of loading with an extremely low stress level. This would occur if the crack growth increment in a cycle is equal to the critical crack length minus the current crack length, i.e.  $\frac{dc}{dN}|_t = c_{\text{crit}} - c(t)$ . Conversely, there is a finite probability that the crack growth rate, for a given sample, will be very small or zero under an extremely high applied stress level. These probabilities are small but nonzero, and therefore these models may not always accurately represent the physics of the crack growth process. More importantly, the assumption that the shaping filter which models the nonwhite auxiliary stochastic process,  $x(\omega, t)$ , is a stationary random process is violated. This is evident from Fig. 1 where  $R_{xx}(t + \theta, t)$  is determined by numerically computing the crack growth rates from experimental data.<sup>2</sup> Although the numerical derivatives of crack length computed from experimental data are noisy to some extent, the observation that

$R_{xx}(t + \theta, t)$  is strongly dependent on the time parameter,  $t$ , leads to the conclusion that  $x(\omega, t)$  is not likely to be a stationary random process. If  $x(\omega, t)$  was indeed stationary,  $R_{xx}(t, t)$  would be a constant for all  $t$  and  $R_{xx}(t + \theta, t)$  would be an identical function of  $\theta$  for all values of  $t$ . Neither of these conditions are satisfied as seen in Fig. 1 and, therefore, the assumption that  $x(\omega, t)$  is stationary cannot be justified. This apparently erroneous assumption possibly contributes to the degraded quality of the standard deviation prediction of crack length near the end of the time frame under consideration by Ray and Tangirala.<sup>17</sup> Furthermore, the trend of the standard deviation predictions is inaccurate as is the shape of the density function predictions reported by Enneking<sup>16</sup> and Spencer and Tang.<sup>9</sup> These observations suggest that an alternative formulation that does not model  $x(\omega, t)$  as a stationary random process may provide more accurate predictions. One such model is presented in the next section.

### 2.1 Lognormal distributed crack length (LDCL) model

An alternative approach to stochastic modeling of fatigue damage is proposed in this paper. This model is based on the assumption that crack length, as opposed to crack growth rate, is lognormal distributed. The resulting lognormal distributed crack length (LDCL) model can be experimentally verified by showing that either crack length measurements are approximately lognormal distributed or the natural logarithm of crack length is approximately Gaussian. Specifically, this approach deals with experimental data of the measured crack length instead of the derived values of crack growth rate and, therefore, allows application of standard statistical tests to examine validity of this assumption. This model, as derived below, eliminates the need to solve computationally expensive stochastic differential equations and thereby requires only the solution of the deterministic fatigue crack growth differential equation (1), (2) or (3).

The major concern in assuming lognormal distribution of the crack length is that the probability of the nonphysical event having a negative crack growth increment is nonzero. This is in contrast to the probability of having infinite or zero crack growth rates with an arbitrary load in the LDCG model as discussed earlier. Such problems are unavoidable whenever the lognormal distribution is used to model the statistics of crack length or crack growth rate. However, as shown below, the purpose of this model formulation is only to predict the standard deviation of crack length either as a function of time, or as a function of cycles, in conjunction with the expected value obtained by the deterministic model of fatigue crack growth. This approach allows construction of the two-parameter lognormal distribution function to describe the statistics of crack length as delineated below.

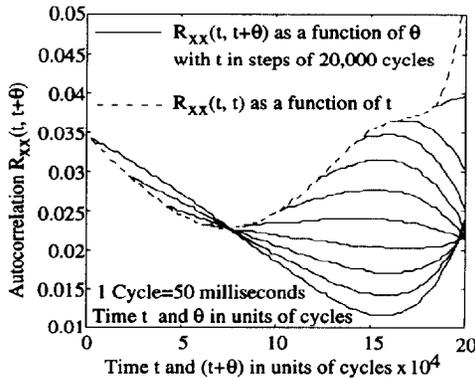


Fig. 1. Profile of autocorrelation  $R_{xx}(t, t + \theta)$ .

We postulate the fatigue-induced crack length,  $c(\omega, t)$ , to be a lognormal distributed stochastic process which is defined as

$$c(\omega, t) = \mu_c(t) \exp\left(z(\omega, t) - \frac{\sigma_z^2(t)}{2}\right) \quad \forall t \geq t_0 \quad (6)$$

where  $t$  indicates (discrete) time in units of cycles;  $\mu_c(t)$  represents the expected value of crack length; and  $\mu_c(t) > 0 \quad \forall t \geq t_0$ . Based on the mean and variance of lognormal distributed random variables, (6) implies that

$$E[c(\omega, t)] = \mu_c(t) \text{ and} \\ \sigma_c(t) = \mu_c(t) \sqrt{\exp(\sigma_z^2) - 1} \quad \forall t \geq t_0 \quad (7)$$

where  $z(\omega, t) \sim N(0, \sigma_z^2(t)) \quad \forall t \geq t_0$  is the zero-mean process defined as

$$z(\omega, t) := \ln\left(\frac{c(\omega, t)}{\mu_c(t)}\right) - E\left[\ln\left(\frac{c(\omega, t)}{\mu_c(t)}\right)\right] \quad (8)$$

It follows from (8) that the following relation:

$$E\left[\ln\left(\frac{c(\omega, t)}{\mu_c(t)}\right)\right] = -\frac{\sigma_z^2(t)}{2} \\ = -\frac{1}{2} \text{Var}\left[\ln\left(\frac{c(\omega, t)}{\mu_c(t)}\right)\right] \quad \forall t \geq t_0 \quad (9)$$

should be satisfied for validity of the postulation in eqn (6). The results generated from (9) are in close agreement with the statistical data of Virkler *et al.*<sup>2</sup> and Ghonem and Dore<sup>3</sup> for 2024-T3 and 7075-T6 aluminum alloys, respectively, at different levels of constant cyclic stress amplitude as reported by Tangirala.<sup>19</sup> It is noted, however, that satisfying eqn (9) is a necessary (but not a sufficient) condition for the crack growth process to be lognormal distributed. Results of other statistical tests, namely, Chi-square and Kolmogorov–Smirnov tests, do not contradict the postulation of eqn (6).

The crack propagation process is an explicit function of the continuous variable  $\mu_c(t)$  and is only implicitly dependent on the parameter  $t$  as seen in (6). Therefore, it is appropriate to model the stochastic processes,  $c(\omega, t)$  and  $z(\omega, t)$ , and their mean square derivatives with a continuous function of  $\mu_c(t)$  as the independent variable in lieu of time,  $t$ . To this effect, we introduce a function of the average crack length,  $\mu_c(t)$ , as

$$\tau(t) := \ln\left(\frac{\mu_c(t)}{\mu_c(t_0)}\right) \quad \forall t \in [t_0, T), \quad t < \infty \quad (10)$$

The dimensionless parameter  $\tau$ , which is a monotonically increasing continuous function of the (discrete) time parameter,  $t$ , is used as the independent variable in the sequel.

In view of the definition in (10), the Gaussian process  $\{z(\omega, t): t \geq t_0\}$  is denoted as  $\{z_\tau: \tau \geq 0\}$ . It is postulated that the random variable  $z_0 = z_{\tau|_{\tau=0}}$  is independent of the random process  $\{z_\tau - z_0: \tau > 0\}$  based on

the rationale that, after an inspection of a structural component, the mean  $\mu_c(t_0)$  and variance  $\sigma_c^2(t_0)$  of the crack length at time  $t_0$  (i.e. at  $\tau(t) = 0$ ) are obtained in terms of the measurement data and precision of the measuring instrument. Subsequently, as the structural component undergoes load cycles (i.e. for  $\tau(t) < 0$ ), the crack growth process is subjected to material-dependent uncertainties which are independent of measurement uncertainties.

Statistical information for  $\{z_\tau: \tau \geq 0\}$  has been generated from each of the four sets of experimental data, namely, one data set of Virkler *et al.*<sup>2</sup> and three data sets of Ghonem and Dore.<sup>3</sup> The profiles of the autocorrelation function  $R_{zz}$  is shown in the plots of Fig. 2 for all four data sets. The autocorrelation function  $R_{zz}$  is observed to approximately obey the following relationship for each of the four sets of fatigue test data:

$$R_{zz}(\tau + \theta, \tau) = R_{zz}(\tau, \tau) + k(\tau)\theta \\ \text{for } \tau \geq 0 \text{ and } (\tau + \theta) \geq 0 \quad (11)$$

where  $k(\cdot)$  is a monotonically increasing continuous function with  $k(0) = 0$ .

Since  $z_\tau$  is a zero-mean process with a continuous covariance function, it can be expressed via the Karhunen–Loève (K–L) expansion.<sup>24</sup> Let the continuous process  $z_\tau$  be discretized at  $m$  points beyond  $\tau = 0$  as  $z = [z_1 - z_0 \quad z_2 - z_0 \quad \dots \quad z_m - z_0]^T$  where  $z_j := z_{\tau_j}$ ,  $j = 1, 2, \dots, m$ . The covariance matrix  $R_{zz} \in \mathcal{R}^{m \times m}$  can be expressed as  $\Phi \Lambda \Phi^T$  with  $\Phi^T \Phi = I$  and  $\Lambda = \text{diag}[\lambda_1 \dots \lambda_m]$ . The deterministic matrix  $\Phi$  is composed of columns,  $\phi^j$ , which are the eigenvectors of  $R_{zz}$ . Therefore, the Gaussian random vector  $z$  can be defined as

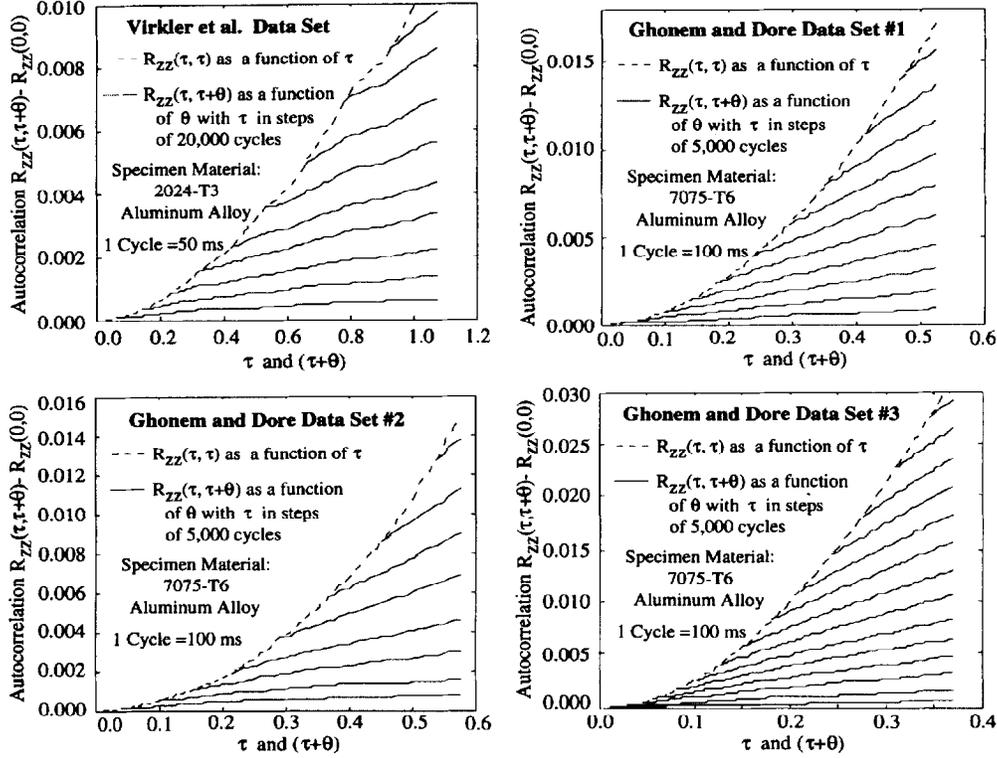
$$z := \Phi X \quad \text{where } \text{Cov}(X) = \Lambda = \text{diag}[\lambda_1 \dots \lambda_m] \quad (12)$$

Since  $z_\tau$  is postulated to be a Gaussian process, orthogonality of the K–L expansion ensures that the random vector  $X = \{X_j \sim N(0, \lambda_j), j = 1, 2, \dots, m\}$  is a set of independent random variables. This leads to

$$z_k = z_0 + \sum_{j=1}^m (\phi_k^j \sqrt{\lambda_j} \chi_j), \quad k = 1, 2, \dots, m \quad (13)$$

where  $\phi_k^j := \phi^j(\tau_k)$  is the  $k$ th element of the eigenvector  $\phi^j$ ; and  $\chi_i \sim N(0, 1), i = 1, 2, \dots, m$  are independent and identically distributed with  $X_i = \sqrt{\lambda_i} \chi_i$ . It is a well-known fact<sup>24</sup> that if the  $m$  eigenvalues of  $R_{zz}$  are ordered as  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_m$ , an approximation  $\hat{z}_k(M) := \hat{z}_{\tau_k}(M)$  of the random variable  $z_k := z_{\tau_k}$ , with  $\hat{z}_0 := z_0$ , is obtained from the first  $M$  eigenvectors as

$$\hat{z}_k(M) := z_0 + \sum_{j=1}^M \phi_k^j \sqrt{\lambda_j} \chi_j \\ k = 1, 2, \dots, m \text{ and } 1 \leq M < m \quad (14)$$


 Fig. 2. Profiles of  $R_{zz}(\tau, \tau + \theta)$ .

having the minimum mean square error

$$\bar{\varepsilon}^2(M) = E \left[ (z - \hat{z}(M))^T (z - \hat{z}(M)) \right] = \sum_{j=M+1}^m \lambda_j \quad (15)$$

Since  $z_0$  is independent of  $\{z_\tau - z_0: \tau > 0\}$  as discussed earlier,  $z_0$  must be independent of each of  $\chi_j$ ,  $j = 1, 2, \dots, m$  and therefore is also independent of each of  $\hat{z}_j - z_0$ ,  $j = 1, 2, \dots, M$  for  $1 \leq M < m$ . Using the statistical orthonormality (i.e. independence) property of  $\chi_j$ ,  $j = 1, 2, \dots, M$ , it follows that the covariance of the approximated random process  $\hat{z}_\tau$  can be obtained as

$$\begin{aligned} R_{\hat{z}\hat{z}}(\tau_k, \tau_l) - R_{\hat{z}\hat{z}}(0, 0) &= \text{Cov}(\hat{z}_k - z_0, \hat{z}_l - z_0) \\ &= \sum_{j=1}^M \lambda_j \phi_k^j \phi_l^j \end{aligned} \quad (16)$$

The statistical results derived from each of the four sets of discrete experimental data<sup>2,3</sup> show that the eigenvector corresponding to the largest eigenvalue,  $\lambda_1$ , can be fitted as a continuous function of  $\tau$

$$\phi^1(\tau) = \left( \sqrt{Q/\lambda_1} \right) \tau \quad (17)$$

and the mean square error in (15) is in the range of 2–5% if  $M$  is chosen to be 1, i.e. if only the first (largest)

eigenvalue is considered. In that case, (16) is modified as

$$R_{\hat{z}\hat{z}}(\tau + \theta, \tau) - R_{\hat{z}\hat{z}}(0, 0) \approx Q\tau(\tau + \theta) \quad (18)$$

where  $R_{\hat{z}\hat{z}}(0, 0)$  is obtained from (7) in terms of the mean,  $\mu_c(t_0)$  and variance,  $\sigma_c^2(t_0)$ , of the crack length.

$$R_{\hat{z}\hat{z}}(0, 0) = R_{zz}(0, 0) = \ln(1 + \sigma_c^2(t_0)/\mu_c^2(t_0)) \quad (19)$$

A comparison of (11) and (18) leads to the following conclusion:

$$\begin{aligned} R_{zz}(\tau, \tau) &\approx R_{\hat{z}\hat{z}}(\tau, \tau) \approx R_{zz}(0, 0) + Q\tau^2 \\ \text{and } k(\tau) &\approx Q\tau \end{aligned} \quad (20)$$

*Remark.* The Karhunen–Loève expansion may be used directly with the crack length to express the covariance of crack length as a function of  $\mu_c(t)$ . The development parallels the formulation described above and is supported by experimental data in that  $R_{\hat{c}\hat{c}}(\mu, \mu)$  and  $R_{\hat{c}\hat{c}}(\mu + \nu, \mu)$  could be approximated by continuous functions of  $(\mu_c(t) - \mu_c(t_0))^2$  and  $(\mu_c(t) - \mu_c(t_0))\nu$  for  $\nu > 0$ , respectively. The reason for using the natural log of crack length via the auxiliary stochastic process,  $z_\tau$ , instead of the actual crack length itself is that the quality of the linear fit for  $R_{\hat{c}\hat{c}}(\mu + \nu, \mu)$  is not as good as that for  $R_{\hat{z}\hat{z}}(\tau + \theta, \tau)$  in eqn (18). This leads to larger prediction errors if the first eigenvector is approximated by a linear function of  $(\mu_c(t) - \mu_c(t_0))^2$ .

Based on the physical understanding of the crack growth and the analysis of experimental data, the following conditions are postulated as:

$$\frac{\partial^2 E[z_{\tau_1} z_{\tau_2}]}{\partial \tau_1 \partial \tau_2} \text{ exists } \forall \tau_1 = \tau_2 = \tau \text{ and } k(\tau)$$

is twice continuously differentiable with  $k(0) = 0$ .

The above postulation and (11) lead to the following conclusion:

$$\frac{\partial^2 R_{zz}(\tau + \theta, \tau)}{\partial \theta \partial \tau} = \frac{\partial^2 R_{zz}(\tau + \theta, \tau)}{\partial \tau \partial \theta} = \frac{dk(\tau)}{d\tau} \quad \forall \tau, \theta \geq 0 \quad (21)$$

This implies that  $\frac{\partial^2 R_{zz}(\tau_1, \tau_2)}{\partial \tau_1 \partial \tau_2}$  exists at each  $\tau_1 = \tau_2 = \tau$  which is a necessary and sufficient condition for the existence of the mean square derivative,  $\dot{z}_\tau$ , of the random process  $z_\tau$  at each  $\tau > 0$ . Therefore, it follows from (21) that

$$R_{zz}(\tau, \tau) = R_{\dot{z}\dot{z}}(\tau, \tau) = k(\tau) \quad \forall \tau > 0 \quad (22)$$

Furthermore, since  $z_\tau$  is a zero-mean Gaussian process

$$\frac{d}{d\tau}(z_\tau^2) = 2\dot{z}_\tau z_\tau \text{ in the m.s. sense} \quad (23)$$

which, by application of the Schwarz inequality, yields

$$E\left[\frac{d}{d\tau}(z_\tau^2)\right] = 2E[\dot{z}_\tau z_\tau] \quad (24)$$

It follows from (22) and (24) that

$$\begin{aligned} \frac{d}{d\tau}(R_{zz}(\tau, \tau)) &= 2k(\tau) \Rightarrow R_{\dot{z}\dot{z}}(\tau, \tau) \\ &= R_{zz}(0, 0) + 2 \int_0^\tau k(\theta) d\theta \end{aligned} \quad (25)$$

Equation (25) along with the application of a standard theorem of mean square calculus<sup>18</sup> yields

$$\begin{aligned} R_{\dot{z}\dot{z}}(\tau + \theta, \tau) &= \frac{\partial^2 R_{zz}(\tau + \theta, \tau)}{\partial \theta \partial \tau} \\ &= \dot{k}(\tau) = R_{\dot{z}\dot{z}}(\tau, \tau) \quad \forall \tau, \theta \end{aligned} \quad (26)$$

Since  $z_\tau$  is Gaussian, its mean square derivative  $\dot{z}_\tau$  is also Gaussian. Hence, (26) implies that  $\dot{z}_\tau$  is an independent increment process and that  $\dot{z}_\tau$  can be regarded as a (possibly varying diffusion) Brownian motion process with

$$E[(\dot{z}_{\tau+\theta} - \dot{z}_\tau)^2] = E[\dot{z}_{\tau+\theta}^2] - E[\dot{z}_\tau^2] \quad (27)$$

Alternatively, (26) can be rewritten as

$$d\dot{z}_\tau = \sqrt{\dot{k}(\tau)} d\beta_\tau \quad (28)$$

where  $\beta_\tau$  is the unit diffusion Brownian motion process.

The above results are expressed in the state-variable setting as

$$\begin{aligned} d \begin{bmatrix} z_\tau \\ \dot{z}_\tau \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} z_\tau \\ \dot{z}_\tau \end{bmatrix} d\tau + \begin{bmatrix} 0 \\ \sqrt{\dot{k}(\tau)} \end{bmatrix} d\beta_\tau \\ E \begin{bmatrix} z_0 \\ \dot{z}_0 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \text{Cov} \begin{bmatrix} z_0 \\ \dot{z}_0 \end{bmatrix} = \begin{bmatrix} \sigma_{z_0}^2 & 0 \\ 0 & \dot{k}(0) \end{bmatrix} \end{aligned} \quad (29)$$

where  $k(0) = 0$  and  $\dot{k}(0)$  is known; and the variance of  $z_\tau$  at  $\tau = 0$  is obtained from (19) as  $\sigma_{z_0}^2 = R_{zz}(0, 0)$ . Equation (29) can be simplified by using the experimentally-observed approximation  $k(\tau) \approx Q\tau$  of (20) as  $dz_\tau = \vartheta d\tau$  under constant amplitude load where the Gaussian random variable  $\vartheta \sim N(0, Q)$  is independent of  $z_0$ , i.e. the process  $z_\tau$  at  $\tau = 0$ . A physical interpretation of the governing equations of the LDCL model is presented in the following remark.

*Remark.* As a structural component is subjected to load cycles, the mean crack length,  $\mu_c(t)$  is obtained via the deterministic model of crack growth in (1), (2) or (3);  $\tau(t)$  and  $\sigma_{z_0}^2$  are obtained from eqns (10) and (19), respectively. The impact of mechanical load (e.g. stress amplitude and peak stress) on the uncertainties of the crack growth behavior is realized by the variance  $Q$  of the random variable  $\vartheta$ . For constant amplitude loading,  $Q$  remains constant. For variable amplitude loading (which has not yet been experimentally verified),  $Q$  varies with the load amplitude, and hence the random variable  $\vartheta$  becomes a stochastic process as depicted in the stochastic differential eqn (29).

### 3 VERIFICATION AND DISCUSSION OF MODEL RESULTS

The stochastic damage model is tuned using the statistical data of fatigue crack growth collected by Virkler *et al.*<sup>2</sup> and Ghonem and Dore<sup>3</sup> for 2024-T3 and 7075-T6 aluminum alloys, respectively, in which the tests were conducted under different constant load amplitudes at ambient temperature. The Virkler data set was generated at a single load level with peak nominal stress of 60.33 MPa (8.75 ksi) and stress ratio  $R = 0.2$ . Ghonem and Dore data was generated at three different load levels: (i) Set 1 with peak nominal stress of 141.30 MPa (20.49 ksi) and  $R = 0.6$ ; (ii) Set 2 with peak nominal stress of 137.95 MPa (20.00 ksi) and  $R = 0.5$ ; and (iii) Set 3 with peak nominal stress of 94.18 MPa (13.66 ksi) and  $R = 0.4$ . Four plots in Fig. 3 display comparisons of model-predicted standard deviation of crack length with the respective statistics based on the four sets of experimental data as a function of load cycle. The model predictions are in close agreement with

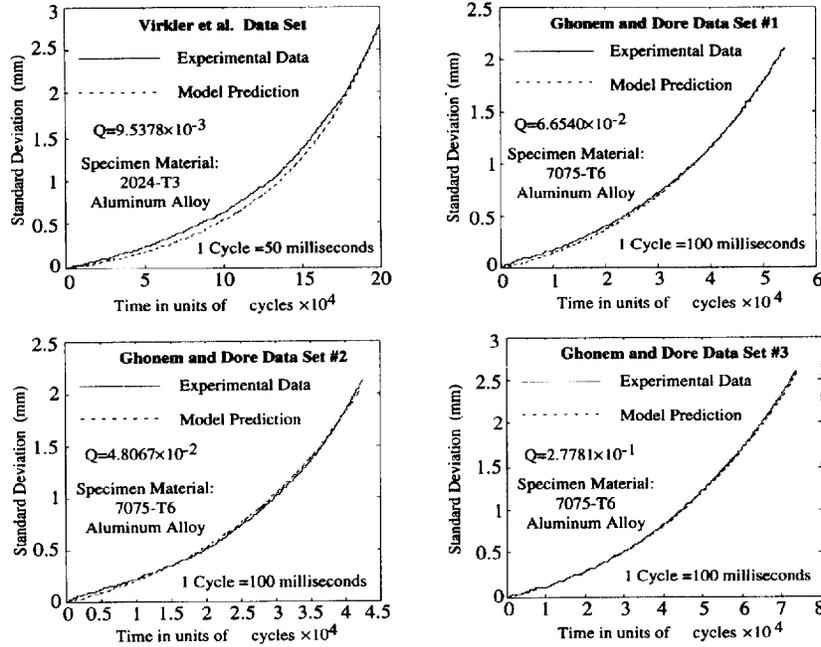


Fig. 3. Standard deviation of crack length for 2024-T3 and 7075-T6 aluminum alloys.

the respective experimental results derived from each of the four data sets. The value of  $Q$ , in all cases, is determined using the methods described in the previous section and yields a close fit of the experimental data. Apparently, the stochastic model parameter  $Q$  is dependent on both material, stress level (i.e. peak stress and  $R$ ) and the initial crack length.

The results, obtained from the lognormal distributed crack length (LDCL) model, are comparable or superior in accuracy to those generated by the lognormal distributed crack growth (LDCG) model of Spencer and Tang,<sup>9</sup> Spencer *et al.*,<sup>10</sup> and Enneking<sup>16</sup> that involved numerical solution of the Kolmogorov diffusion equations to obtain the statistics of crack length. It has been pointed out by Ray and Tangirala<sup>17</sup> that real-time solution of the partial differential equations in the Kolmogorov equations is not computationally feasible even on mainframe computers. In contrast, the LDCL model requires the solution of only a few algebraic identities which can be conveniently executed in real-time on a microcomputer. However, it should be noted that the Kolmogorov equations yield the conditional density as a function of time while the proposed LDCL model, in the present form, generates the two-parameter lognormal distribution function from the first two moments of crack length. The assumption of lognormal distribution of crack length is justified by experimental data. In addition, if the crack growth process is assumed to be perfectly correlated over a finite horizon of crack length, it is possible to model both crack length and crack growth rate as lognormal distributed.

#### 4 SUMMARY AND CONCLUSIONS

The stochastic model of fatigue crack dynamics, presented in this paper, is based on the principle of Gauss–Markov processes and is formulated under the assumption that crack length is lognormal distributed instead of the more common assumption that crack growth rate is lognormal distributed. The standard deviation of crack length is obtained as a solution of algebraic identities that is computationally much faster than solving the Kolmogorov forward equation in the Itô integral setting or the extended Kalman filter equation in the Wiener integral setting. The proposed lognormal distributed crack length (LDCL) model has been verified with experimental fatigue crack growth data of 2024-T3 and 7075-T6 aluminum alloys at different levels of constant amplitude load excitation. Extension of this model to load excitation of varying amplitude is the subject of current research.

The probability distribution of (nonstationary) random crack length can be generated from the first two moments based on the assumption of lognormal distribution. This information suffices to determine, at a given level of confidence, the time-dependent remaining life of mechanical structures subjected to a specified anticipated load profile. The stochastic damage information can be used in hypothesis testing algorithms to generate early warnings of impending failures in real-time and also for making dynamic decisions regarding plant operations and maintenance schedules. In essence, the increased speed of the proposed model for computation

of the statistics of fatigue crack length makes it ideally suited to real-time damage monitoring and failure prognostic applications.

While the stochastic modeling approach, described here, focuses on the inherent material uncertainties, there are two other major sources of uncertainties, namely, random loading effects and unknown initial conditions which have not been considered here. A unified model that accounts for all three primary sources of uncertainties in fatigue crack growth needs to be developed before practical applications become viable.

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