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A stochastic model of fatigue crack propagation under variable-amplitude loading

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Abstract

This paper presents a stochastic model of fatigue crack propagation in ductile alloys that are commonly encountered in mechanical structures and machine components of complex systems (e.g. aircraft, spacecraft, ships and submarines, and power plants). The stochastic model is built upon a deterministic state-space model of fatigue crack propagation under variable-amplitude loading. The (non-stationary) statistic of the crack growth process for center-cracked specimens is obtained as a closed form solution of the stochastic differential equations. Model predictions are in agreement with experimental data for specimens fabricated from 2024-T3 and 7075-T6 aluminum alloys and Ti-6Al-4 V alloy subjected to constant-amplitude and variable-amplitude loading, respectively. The stochastic model of crack propagation can be executed in real time on an inexpensive platform such as a Pentium processor. © 1998 Elsevier Science Ltd. All rights reserved.

Keywords: Fatigue and fracture; Variable-amplitude loading; Stochastic modeling; State-space

1. Introduction

Fatigue crack analysis is an essential tool for life prediction and maintenance of machinery components that are subjected to cyclic stresses over a prolonged period of time [9]. The current state-of-the-art of fatigue crack analysis, especially for high-cycle fatigue, is based largely on the assumption of constant-amplitude loading in a deterministic setting. An

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Nomenclature

a	function coefficients
A	function coefficients
c	crack length
E	Young's modulus
f	function for stress intensity factor
F	geometry factor in the crack growth equation
$K_{\mathrm{I}}^{\mathrm{cc}}$	mode I stress intensity factor (center-cracked)
$K_{\mathrm{I}}^{\mathrm{com}}$	mode I stress intensity factor (compact)
m	exponent in the crack growth equation
$P^{\max}$	maximum load in units of force
$P^{\min}$	minimum load in units of force
S^{flow}	flow stress
$S^{\max}$	maximum stress within a cycle
$S^{\min}$	minimum stress within a cycle
S^{o}	crack opening stress
S^{oss}	steady-state crack opening stress
S^{ult}	ultimate tensile strength
S^{y}	yield stress
t	time in number of cycles
w	width or half-width of test specimens
z	normal random process related to crack growth
α	constraint (plane stress/strain) factor
β	a constant model parameter larger than 1 (≈ 10)
δ	increment operator
ΔS	effective stress range
ϑ^2	constant variance of $\Omega(\zeta, \Delta S)(\Delta S)^m$
η	random decay rate for S^{o}
λ	pulse scaling factor for S^{o}
μ_x	expected value of a random variable X
ρ	material inhomogeneity and measurement noise
σ_x^2	variance of a random variable X
θ	dimensionless parameter c/w
τ	specimen thickness
Ω	random process for crack growth equation
ξ	zero-mean unit variance normal random variable
ψ	functional representation of crack length
ζ	sample point (i.e. a test specimen)

important issue in failure diagnostics and life prediction of operating machinery is uncertainties of the fatigue damage process. Specifically, the standard deviation of predicted time to failure bears an inverse relationship to the amplitude of stress excitation. That is, the predicted service life under high-cycle fatigue suffers from larger uncertainties than that under low-cycle fatigue. This fact is established by experimental investigations including the tests of [4] on specimens of 7075-T6 aluminum alloys at different levels of constant-amplitude loading. Furthermore, over the entire service life, machinery components could be subjected to variable-amplitude loading beyond what is expected under normal operating conditions of constant-amplitude loading. To this effect, a factor of safety is usually taken into consideration to accommodate such incidents. This design approach is often over-conservative because the effects of stress overload may not always be harmful and, under certain conditions (e.g. crack retardation), turn out to be beneficial. On the other hand, the so-called *safe design for infinite life* may be overly optimistic from the point of view of machinery safety and reliability.

The objective here is to analyze the uncertain dynamics of fatigue crack propagation under both constant-amplitude and variable-amplitude loading, as needed for failure diagnostics and health monitoring of operating machinery [9]. This paper presents a stochastic model of fatigue crack propagation in ductile alloys which are commonly encountered in mechanical structures and machine components of complex systems (e.g. aircraft, spacecraft, ships and submarines, and power plants). The stochastic model is built upon the framework of a deterministic fracture-mechanics-based state-space model [11] by randomizing pertinent parameters as well as by incorporating additional stochastic parameters [13]. The purpose of the state-space model is to capture the the phenomena of crack retardation and sequence effects under variable-amplitude loading where the state variables are the crack length and the crack opening stress. The non-stationary statistic of the crack growth process in center-cracked specimens under (tensile) variable-amplitude loading is obtained as a closed form solution of the stochastic differential equations. Subsequently, this model is modified for compact specimens. Model predictions are in agreement with the fatigue test data for center-cracked specimens made of 2024-T3 and 7075-T6 aluminum alloys at different constant-amplitude loading and for compact specimens made of Ti-6Al-4 V alloy under both constant-amplitude and variable-amplitude loading. The stochastic crack propagation model can be executed in real time on an inexpensive platform such as a Pentium processor.

2. Fatigue crack propagation model for center-cracked specimens

The stochastic model of fatigue crack propagation, presented in this paper, is built upon the deterministic state-space model [11] with two state variables, crack length and crack opening stress. Ditlevsen [2] has shown that, under constant-amplitude loading, randomness of fatigue crack propagation accrues primarily from parametric variations. The stochastic process of crack growth rate is largely dependent on two random variables, namely, a multiplicative parameter and an exponent parameter, which are constants for a given specimen (i.e. a given sample point ζ). Ditlevsen [2] has also suggested the possibility of one of the two random variables being a constant for all specimens. Statistical analysis of the experimental data for 2024-T3 and 7075-T6 aluminum alloys [13] shows that the random exponent can be approximated as a constant for all specimens at different levels of constant-amplitude loading for a given material. Based on this observation and the structure of the deterministic state-space model [11], we postulate the following governing equation for fatigue crack propagation in a stochastic setting, partially similar to what was originally proposed by [10] for a deterministic model:

$$\delta c(\zeta,t) = \Omega(\zeta,\Delta S(\zeta,t))(\Delta S(\zeta,t))^m c(\zeta,t)^{m/2}(F(c(\zeta,t)))^m \rho(\zeta,t)\delta t;\ \text{given } c(\zeta,t_0) \text{ and } t \geq t_0 \tag{1}$$

where ζ indicates a given test specimen or a machinery component; $c(\zeta,t)$ is the sum of the measured crack length and the plastic zone radius of the test specimen ζ at the end of a cycle at time t; $\delta c(\zeta,t) \equiv c(\zeta,t+\delta t) - c(\zeta,t)$ is the crack length increment of the test specimen ζ during the cycle beginning at time t and ending at time t, δt; the random process $\Omega(\zeta,\Delta S)$ represents uncertainties of a given ζ at a specific stress range ΔS (i.e. Ω is a constant for a given specimen under constant-amplitude stress); the exponent, $m(\zeta) = m$ w.p. 1, is a material parameter and greater than 2 and less than 4 for ductile alloys ([15], p. 193); the noise process $\rho(\zeta,t)$ represents the uncertainties in the material microstructure and crack length measurements that vary with t as the crack propagates even for the same specimen ζ. The multiplicative uncertainty $\rho(\zeta,t)$ in the crack growth process in Eq. (1) is assumed to be a white (i.e. perfectly uncorrelated) stationary process that is statistically independent of $\Omega(\zeta, \Delta S)$. The rationale for this assumption is that inhomogeneity of the material microstructure and measurement noise associated with each test specimen, represented by $\rho(\zeta,t)$, are statistically homogeneous and are unaffected by the uncertainty, $\Omega(\zeta,\Delta S)$, of a particular specimen caused by, for example, machining operations. Without loss of generality, we set $\mu_\rho \equiv E[\rho(\zeta,t)] = 1$ via appropriate scaling of the parameter $\Omega(\zeta,\Delta S)$.

To realize the effects of overload and underload in the stochastic model, the crack opening stress is modeled as a random process $S^o(\zeta,t)$ by converting the decay rate η in the deterministic model of [11] to a random parameter $\eta(\zeta)$. Specifically, the decay rate $\eta(\zeta^*)$ is invariant under all load spectra for a given sample $\zeta*$. This phenomenon has been experimentally observed by several investigators including [12] and [5]. The governing equations for the transient behavior of the stochastic process $S^o(\zeta t)$ are presented below (see the Nomenclature):

$$S^o(\zeta,t) = \left(\frac{1}{1+\eta(\zeta)}\right) S^o(\zeta,t-\delta t) + \left(\frac{\eta(\zeta)}{1+\eta(\zeta)}\right) S^{oss}(t) \quad \text{if } S^{oss}(t) \leq S^o(\zeta,t-\delta t)$$

$$S^o(\zeta,t) = \left(\frac{1}{1+\eta(\zeta)}\right) S^o(\zeta,t-\delta t) + \left(\frac{\eta(\zeta)}{1+\eta(\zeta)}\right) S^{oss}(t) + \left(\frac{\lambda(t)}{1+\eta(\zeta)}\right)\left(S^{oss}(t) - S^o(\zeta,t-\delta t)\right) \quad \text{otherwise} \tag{2}$$

where

$$\lambda(t) = \frac{S^{max}(t) - S^{mod}(t)}{S^{max}(t) - S^{min}(t-\delta t)}$$

having

$$S^{\text{mod}}(t) = \frac{\alpha S^{\text{min}}(t) + S^{\text{min}}(t - \delta t)}{\alpha + 1}$$

and α is the constraint factor (1 for plane stress and 3 for plane strain). The forcing function $S^{\text{oss}}(t)$ at the tth cycle matches the crack opening stress derived from the following semi-empirical relation in SI units for metallic materials [8]:

$$\frac{S^{\text{oss}}(t)}{S^{\text{max}}(t)} = A_0 + A_1 R(t) + A_2 R(t)^2 + A_3 R(t)^3$$

$$R(t) = \frac{S^{\text{mod}}(t)}{S^{\text{max}}(t)}$$

$$S^{\text{mod}}(t) = \frac{\alpha S^{\text{min}}(t) + S^{\text{min}}(t - \delta t)}{\alpha + 1}$$

$$\lambda(t) = \frac{S^{\text{max}}(t) - S^{\text{mod}}(t)}{S^{\text{max}}(t) - S^{\text{min}}(t - \delta t)}$$

$$A_0 = \left(0.825 - 0.34\alpha + 0.005\alpha^2\right)\left[\cos\left(\frac{\pi}{2}\frac{S^{\text{max}}(t)}{S^{\text{flow}}}\right)\right]^{1/\alpha}$$

$$A_1 = (0.415 - 0.071\alpha)\left(\frac{S^{\text{max}}(t)}{S^{\text{flow}}}\right)$$

$$A_2 = 1 - A_0 - A_1 - A_3 \quad \text{if } R(t) > 0$$

$$A_2 = 0 \quad \text{if } R(t) \leq 0$$

$$A_3 = 2A_0 + A_1 - 1 \quad \text{if } R(t) > 0$$

$$A_3 = 0 \quad \text{if } R(t) \leq 0 \tag{3}$$

The randomized decay rate parameter $\eta(\zeta)$ in the stochastic model is made independent of both $\Omega(\zeta,\Delta S)$ and $\rho(\zeta,t)$ based on the experimental observations [5]. The probability distribution function of the random variable $\eta(\zeta)$ is identified from test data. For compatibility with the deterministic model [11], $\eta(\zeta)$ must be dependent on the thickness, τ, and half-width (or width), w, of the test specimen, the yield strength, S^{y}, ultimate tensile strength, S^{ult}, and Young's modulus, E, of the specimen material.

The difference Eqs. (1) and (2), supported by the algebraic Eq. (3), govern the dynamics of crack propagation. Since the numerical solution of these nonlinear stochastic difference Eq. (1) is computationally intensive [7], the objective here is to obtain a closed form solution that can be obtained in real time. Since the time (cycle)-scale of crack length dynamics in Eq. (1) is slow relative to that of crack opening stress dynamics in Eq. (2), the fatigue crack propagation can be treated as a two-time-scale problem. The effective stress range $\Delta S(\zeta,t) \equiv \max(0, (S^{\max}(t) - S^{\mathrm{o}}(\zeta,t)))$, generated from Eq. (2), becomes an instantaneous (i.e. a cycle-dependent) input to Eq. (1). Statistics of random parameters for crack growth are now developed based on fatigue test data for center-cracked specimens.

3. Parameter identification and model verification for center-cracked specimens

Given the geometry factor

$$F = \sqrt{\sec\left(\frac{\pi}{2}\frac{c(\zeta,t)}{w}\right)}$$

for center-cracked specimens, the crack growth rate in Eq. (1) can be separated into terms involving $c(\zeta,t)$ and $\Delta S(\zeta,t)$. Approximation of the first two terms in the Taylor series expansion of the secant term in Fyields the following stochastic differential equation:

$$\left((c(\zeta,t))^{-m/2} - m\left(\frac{\pi}{4w}\right)^2 (c(\zeta,t))^{2-m/2}\right)\delta c(\zeta,t) = \Omega(\zeta,\Delta S(\zeta,t))(\Delta S(\zeta,t))^m \rho(\zeta,t)\delta t \tag{4}$$

Note that the error due to truncation of the Taylor series expansion in Eq. (4) monotonically increases with $c(\zeta,t)$. Nevertheless, for crack length region of interest in engineering applications, this error is insignificant.

Since the number of cycles to failure is usually very large in the crack propagation processes (even for low-cycle fatigue), a common practice in the fracture mechanics literature is to approximate the difference equation of crack propagation by a differential equation. In essence, the Riemann sum resulting from the solution of the difference Eq. (4) is replaced by an integral. Therefore, Eq. (4) is approximated as a stochastic differential equation with input excitation $\Delta S(\zeta,t)$ which is integrated pointwise (i.e. for the individual ζs) to obtain:

$$\psi(\zeta,t,t_0) = \int_{t_0}^{t} \mathrm{d}\tau \Omega\big(\zeta,\Delta S(\zeta,\tau)(\Delta S(\zeta,\tau))^m\big)\rho(\zeta,t) \tag{5}$$

where

$$\psi(\zeta,t,t_0) \equiv \left(\frac{c(\zeta,t)^{1-m/2} - c(\zeta,t_0)^{1-m/2}}{1-\dfrac{m}{2}}\right) - m\left(\frac{\pi}{4w}\right)^2\left(\frac{c(\zeta,t)^{3-m/2} - c(\zeta,t_0)^{3-m/2}}{3-\dfrac{m}{2}}\right) \tag{6}$$

Note that, for ductile alloys, the constant parameter m is greater than 2 and less than 4. Hence it is guaranteed that Eq. (6) is valid.

The model parameters and their statistics are identified from experimental data of random fatigue crack propagation for center-cracked specimens of 2024-T3 aluminum alloy [16] and 7075-T6 aluminum alloy [4]. The Virkler specimens were of half-width $w = 76.2$ mm, thickness $\tau = 2.54$ mm and half-notch length = 1.27 mm and the Ghonem specimens were of half-width $w = 50.8$ mm, thickness $\tau = 3.175$ mm and notch half-length = 7.144 mm, in which the tests were conducted under different constant load amplitudes at ambient temperature. The Virkler data set was generated from 68 specimens at a single constant-amplitude load with peak nominal stress of 60.33 MPa (8.75 ksi) and stress ratio $R \equiv S^{\min}/S^{\max} = 0.2$ for 200,000 cycles. The Ghonem data sets were generated from 60 specimens each at three different levels of constant-amplitude loading as: (i) set 1 with the peak nominal stress of 70.65 MPa (10.25 ksi) and $R = 0.6$ for 54,000 cycles; (ii) set 2 with the peak nominal stress of 69.00 MPa (10.00 ksi) and $R = 0.5$ for 42,350 cycles; and (iii) set 3 with the peak nominal stress of 47.09 MPa (6.83 ksi) and $R = 0.4$ for 73,500 cycles.

The exponent parameter m is first identified as the ensemble average estimate from the slope of the logarithm of crack growth rate based on the data sets of constant-amplitude loading. The estimated values of m are 3.4 and 3.6 for 2024-T3 and 7075-T6 aluminum alloys, respectively. In the above four cases of constant-amplitude loading, the steady-state values of $\Delta S(\zeta,t)$ are reduced to four different constants as listed in Table 1. Note that the random decay rate $\eta(\zeta)$ in Eq. (2) does not influence the steady-state value of the crack opening stress S^{oss}.

The generalized parametric relations of the fatigue crack propagation model are proposed by [13] based on Karhunen–Loève expansion [3] of the stochastic process $\psi(\zeta,t,t_0)$ in Eq. (6) where the model parameters for 2024-T3 and 7075-T6 aluminum alloys are evaluated from Virkler and Ghonem data sets at different levels of constant-amplitude loading using the crack growth model of [6]. Similar results are generated in this paper based on the state-space model derived in Section 2. The model parameters are listed in Table 1 for 2024-T3 and 7075-T6 aluminum alloys and the pertinent conclusions are summarized below:

$$\mu_\Omega(\Delta S) \equiv E\big[\Omega(\zeta,\Delta S)\big] \text{ is independent of } \Delta S \text{, i.e. } \mu_\Omega \text{ is a constant and} \quad E\big[(\Delta S)^m \Omega(\zeta,\Delta S)\big] = (\Delta S)^m \mu_\Omega \tag{7a}$$

$$\sigma_\Omega^2(\Delta S) \equiv \mathrm{Var}\big[\Omega(\zeta,\Delta S)\big] \text{ is proportional to } (\Delta S)^{-2m} \text{, i.e. } \mathrm{Var}\big[(\Delta S)^m \Omega(\zeta,\Delta S)\big] = \vartheta^2 \text{ is a constant} \tag{7b}$$

$$\mathrm{Var}\left[(\Delta S)^m \int_{t_0}^{t} \mathrm{d}\tau(\rho(\zeta,\tau) - 1)\right] \text{ is small compared to } \mathrm{Var}\big[(\Delta S)^m \Omega(\zeta)(t - t_0)\big] \text{ for large } (t - t_0) \tag{7c}$$

Based on experimental observations under constant-amplitude loading and the short time-scale of ΔS relative to that of crack propagation, we conjecture that the properties (7a)–(7c) also

Table 1
Model parameters for 2024-T3 and 7075-T6 alloys based on Virkler and Ghonem data sets

Data set and material type	Effective stress range $\Delta S \equiv S^{\max} - S^{o}$ (MPa)	m (dimensionless)	μ_Ω (SI units)	$(\Delta S)^m \sigma_\Omega / \mu_\Omega$ (SI units)	μ_ρ (dimensionless)	$(\Delta S)^m \sigma_\rho / \mu_\rho$ (SI units)
Virkler data 2024-T3	31.91	3.4	1.6×10^{-7}	0.875×10^{4}	1.0	5.0990
Ghonem data. no. 1, 7075-T6	23.81	3.6	1.6×10^{-7}	2.8125×10^{4}	1.0	4.6904
Ghonem data, no. 2, 7075-T6	27.58	3.6	1.6×10^{-7}	2.8125×10^{4}	1.0	4.6904
Ghonem data, no. 3, 7075-T6	21.30	3.6	1.6×10^{-7}	2.8125×10^{4}	1.0	4.6904

hold under variable-amplitude loading for a given ensemble of specimens. Following Eq. (5) and assuming negligible σ_ρ^2, the (cycle-dependent) mean and variance of the process $\Omega(\zeta,\Delta S)$ are obtained for any given $t_{\rm f} \gg t_0$ as:

$$\mu_\Omega = \left(\int_{t_0}^{t_f} {\rm d}\tau(\Delta S(\zeta,\tau))^m \right)^{-1} \mu_\psi(t_f,t_0) \text{ and } \vartheta^2 \approx (t_f - t_0)^{-2} \sigma_\psi^2(t_f,t_0) \tag{8}$$

where $\mu_\psi(t_{\rm f},\ t_0)$ and $\sigma_\psi^2(t_{\rm f},t_0)$ are the mean and variance of $\psi(\zeta,t_{\rm f},t_0)$, respectively, in Eqs. (5) and (6). For variable-amplitude loading on the same ensemble of specimens, the parameter σ_Ω^2 is conjectured to remain proportional to $(\Delta S)^{-2m}$, i.e. $\mathrm{Var}\lfloor \Omega(\zeta,\Delta S)(\Delta S)^m \rfloor = \vartheta^2$ is an invariant.

Several investigators have assumed that the crack growth rate in metallic materials is two-parameter log–normal distributed [14] under constant-amplitude loading. We hypothesize that the random process $\Omega(\zeta,\Delta S)$ is two-parameter ($r=2$) log–normal distributed, and its goodness of fit is tested by both χ^2 and Kolmogorov–Smirnov tests. Each of the four data sets is partitioned into $L=12$ segments to assure that each segment contains at least five samples. With $(L-r-1) = 9$ degrees of freedom, the χ^2-test shows that, for each of the four data sets, the hypothesis of two-parameter log–normal distribution of $\Omega(\zeta,\Delta S)$, at different values of ΔS, passed the 10% significance level which suffices the conventional standard of 5% significance level. For each of the four data sets [4, 16], the hypothesis of two-parameter log–normal distribution of $\Omega(\zeta,\Delta S)$ also passed the 20% significance level of the Kolmogorov–Smirnov test.

Now we present the crack length predictions via Monte Carlo simulation of the stochastic crack propagation model along with experimental data subjected to constant-amplitude loading for different values of $R \equiv S^{\min}/S^{\max}$. Fig. 1 shows a comparison of the Virkler and Ghonem data sets in the four plates on the right side with the model predictions in the corresponding four plates on the left side. The log–normal distributions of both $\Omega(\zeta,\Delta S)$ and $\rho(\zeta,t)$ are simulated by taking exponential of the standard normal random number generator. The scatter of the crack growth profile of individual test specimens in the model prediction is primarily due to the random parameter $\Omega(\zeta,\Delta S)$. It should be noted that the randomization of $\Omega(\zeta,\Delta S)$ alone does not account for the complete variability of realization of the crack growth. The random noise process $\rho(\zeta,t)$, which represents material inhomogeneity and measurement noise, causes the individual sample paths to cross each other. Ideally, the sample paths should not cross and be isolated from each other for specimens made of a perfectly homogeneous material and having noiseless measurements.

4. Parameter identification and model verification for compact specimens

The geometry factor F for compact specimens is different from that of center-cracked specimens; the instantaneous stress intensity factor [1] for compact specimens is obtained as:

$$K_{\rm I}^{\rm com} = f\left(\frac{c(\zeta,t)}{w}\right)\sqrt{w}S(t) \tag{9}$$

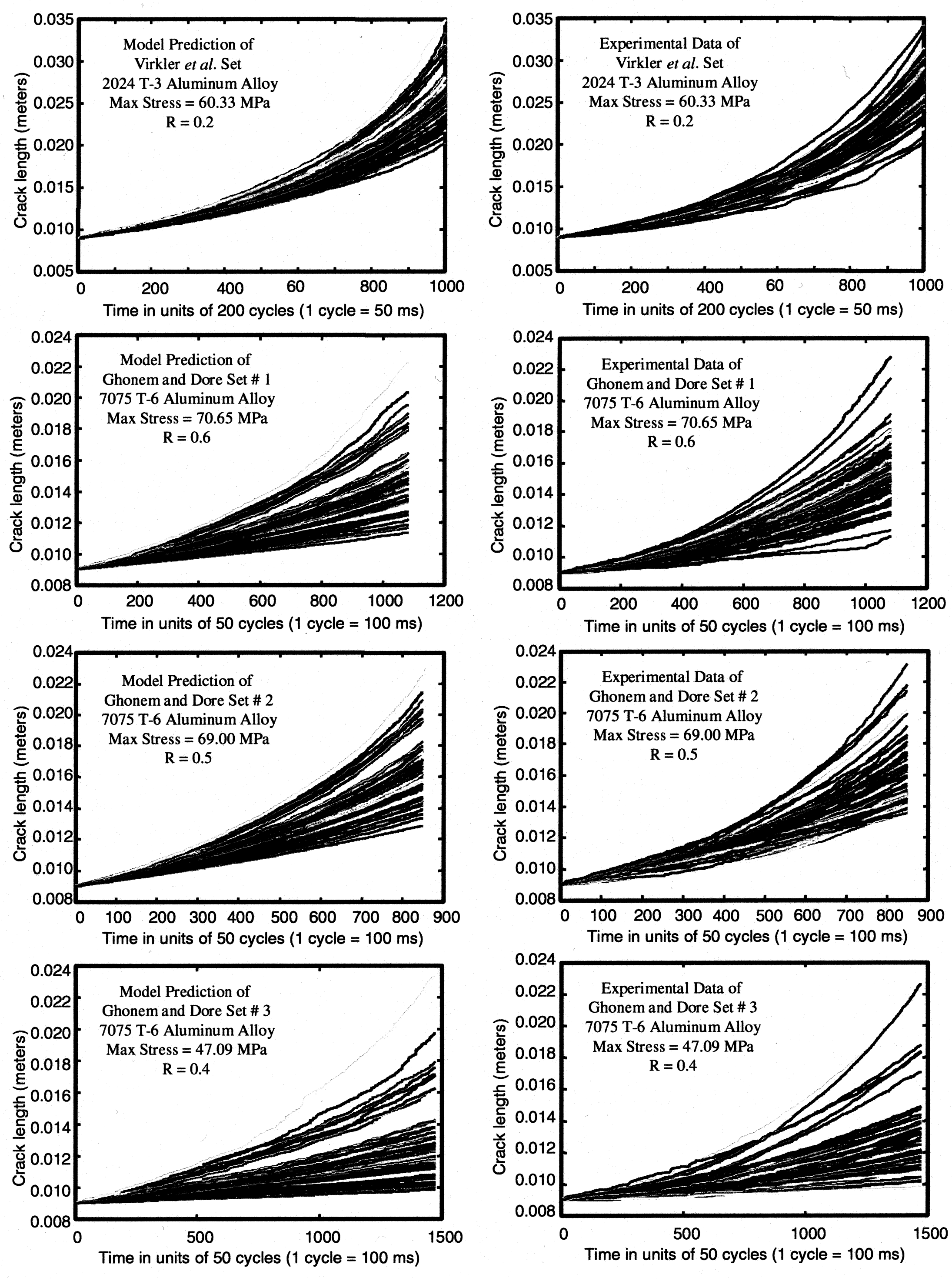

Fig. 1. Model verification with experimental data for center-cracked specimens of 2024-T3 and 7075-T6 alloys.

$$f(\theta) \equiv \frac{2+\theta}{\sqrt{(1-\theta)^3}}\left(\sum_{k=0}^{4} a_k \theta^k\right) \text{ for } 0 \le \theta < 1 \tag{10}$$

where $c(\zeta,t)$ is the crack length process as defined earlier, τ and w are thickness and width of the compact specimen, $S(t) \equiv P(t)/\tau w$ is the nominal stress under the applied load $P(t)$, and the coefficient vector $[a_0\ a_1\ a_2\ a_3\ a_4]$ = [0.886 4.64 −13.32 14.72 −5.60] in SI units.

Let us equate the stress intensity factor of a compact specimen with that of a center-cracked specimen for which:

$$K_{\mathrm{I}}^{\mathrm{cc}} = \sqrt{\pi c(\zeta,t) \sec\left(\frac{\pi}{2}\frac{c(\zeta,t)}{w}\right)} S_{\mathrm{eq}}(t) \tag{11}$$

where

$$S_{\mathrm{eq}}(t) \equiv \frac{P_{\mathrm{eq}}(t)}{\tau w}$$

is the nominal stress under the applied load $P_{\mathrm{eq}}(t)$ for an equivalent center-cracked specimen. In order that $K_{\mathrm{I}}^{\mathrm{cc}}$ in Eq. (11) becomes equal to $K_{\mathrm{I}}^{\mathrm{com}}$ in Eq. (9), the stress cycle $(S_{\mathrm{eq}}^{\max}(t), S_{\mathrm{eq}}^{\min}(t))$ for an equivalent center-cracked specimen is obtained corresponding to a load cycle $(P^{\max}, P^{\min})$ on a compact specimen as:

$$S_{\mathrm{eq}}^{\max}(t) = g\left(\frac{c(\zeta,t)}{w}\right)\frac{P^{\max}(t)}{\tau w} \text{ and } S_{\mathrm{eq}}^{\min}(t) = g\left(\frac{c(\zeta,t)}{w}\right)\frac{P^{\min}(t)}{\tau w} \tag{12}$$

$$g(\theta) \equiv \frac{f(\theta)}{\pi\theta \sec\left(\frac{\pi}{2}\theta\right)} \text{ for } 0 \le \theta < 1 \tag{13}$$

It follows from Eqs. (10), (12) and (13) that $S_{\mathrm{eq}}^{\max}(t)$ and $S_{\mathrm{eq}}^{\min}(t)$ are functions of crack length. Although the stress ratio $S_{\mathrm{eq}}^{\min}(t)/S_{\mathrm{eq}}^{\max}(t)$ is independent of $c(\zeta,t)$, the forcing function $S_{\mathrm{eq}}^{\mathrm{oss}}(t)/S_{\mathrm{eq}}^{\max}(t)$ in the governing Eq. (2) for the crack opening stress is not independent of $c(\zeta,t)$ as seen in Eq. (3). Therefore, unlike Eq. (1) for center-cracked specimens, the crack growth equation for compact specimens may not be separable into a function of the crack length $c(\zeta,t)$ and another function of the instantaneous effective stress range $\Delta S(\zeta,t) \equiv S^{\max}(t) - S^{\mathrm{o}}(\zeta,t)$. Also, unlike Eq. (4) for center-cracked specimens, the stochastic parameter $\Omega(\zeta, \Delta S(\zeta,t))$ cannot be obtained as an explicit function of $c(\zeta,t)$.

For compact specimens, we have used the overload data of random fatigue crack propagation for Ti–6Al–4 V alloy specimens of width $w = 76.2$ mm, thickness $\tau = 7.2$ mm and notch length = 15 mm [5]. Data from 35 out of 60 specimens were found to be suitable for parameter estimation as other specimens are not subjected to the same load excitation. Each data set starts with a crack length of 20 mm and having two overload regions roughly starting at 25 and 29 mm. The crack length at which the overloads are applied to different specimen differ slightly but no overload region interacts with the other. Basic load conditions are:

$P^{\min}=4$ kN and $P^{\max}=9$ kN. Single cycle overload of $P^{o\ell}=18$ kN was applied at 25 and 29 mm crack length and the base constant-amplitude load was continued after application of the single-cycle overload. Each specimen in the [5] data set responded to the application of overload by instantaneous crack tip extension via ductile rupture on a plane inclined to the normal to the load plane. This crack deflection phenomenon is limited to the surface layer where plane stress conditions exist with a thickness of layer ≈ 500 μm. The deflected crack growth during the overload application is caused by ductile rupture and not by fatigue. The phenomenon of ductile rupture causes a massive crack arrest that is not entirely captured by the state-space model of fatigue crack growth model. The state-space model is now modified by redifining the pulse scaling factor $\lambda(t)$ in Eq. (2) as:

$$\lambda(t) = 2\left(\frac{S^{\max}(t) - S^{\mathrm{mod}}(t)}{S^{\max}(t) - S^{\min}(t - \delta t)}\right) \tag{14}$$

to represent this crack arrest phenomenon that is not caused by fatigue alone. The factor of 2 in Eq. (14) is identified as aparameter to fit all 35 specimens under consideration. Note that the original state-space model is capable of representing crack arrest caused by fatigue loading without any correction factor. This has been checked for center-cracked specimens of [12] subjected to single overload excitation by [11].

The exponent parameter, m, is first identified as the ensemble average estimate from the initial part of the data set where a constant-amplitude load was applied while the crack grew from 20 to 25 mm. During this crack growth process, ΔS remained almost constant for the compact specimens. The exponent $m=2.93$ was identified for Ti–6Al–4 V alloy. Knowing the exponent parameter m, the database for the random process $\Omega(\zeta,\Delta S)$ is generated for the (almost) constant ΔS (which is dependent on the crack length) of the experimental data. A procedure for identification of the parameters μ_Ω and ϑ^2 in compact specimens is outlined below.

Let us define a process $z(\zeta,\Delta S)\approx \mathrm{normal}(\mu_z(\Delta S), \sigma_z^2(\Delta S))$ as:

$$z(\zeta,\Delta S) \equiv \ell\mathrm{n}\left(\frac{\Omega(\zeta,\Delta S)}{\mu_\Omega}\right)$$

Since $\Omega(\zeta, \Delta S)$ is log–normal,

$$\sigma_\Omega^2(\Delta S) = \left(\exp\left(\sigma_z^2(\Delta S)\right) - 1\right)\mu_\Omega^2 \text{ and } \mu_\Omega = \exp\left(\mu_z(\Delta S) + \frac{\sigma_z^2(\Delta S)}{2}\right) \tag{15}$$

Based on the observations on constant-amplitude loading, we conjecture that, for variable-amplitude loading on the same ensemble of specimens, the parameter $\sigma_\Omega^2(\Delta S)$ remains proportional to $(\Delta S)^{-2m}$, i.e. $\mathrm{var}[(\Delta S)^m\Omega(\zeta,\Delta S)]=\vartheta^2$ which is a constant. Therefore,

$$z(\zeta,\Delta S)\approx \mathrm{normal}\left(-\ell\mathrm{n}\left(\sqrt{1+\frac{\vartheta^2(\Delta S)^{-2m}}{\mu_\Omega^2}}\right), \ell\mathrm{n}\left(1+\frac{\vartheta^2(\Delta S)^{-2m}}{\mu_\Omega^2}\right)\right) \tag{16}$$

For a sample path (i.e. an individual specimen) $\zeta*$ under a given cycle-dependent ΔS, the parameter $\Omega(\zeta*,\Delta S)$ can be simulated by selecting $\zeta*$ from a random number generator normal (0. 1) as:

$$\Omega(\zeta^*,\Delta S) = \mu_\Omega \exp\big(z(\zeta^*,\Delta S)\big) \tag{17}$$

where

$$z(\zeta^*,\Delta S) = -\ell\mathrm{n}\left(\sqrt{1+\frac{\vartheta^2(\Delta S)^{-2m}}{\mu_\Omega^2}}\right) + \xi(\zeta^*)\sqrt{\ell\mathrm{n}\left(1+\frac{\vartheta^2(\Delta S)^{-2m}}{\mu_\Omega^2}\right)} \tag{18}$$

and the parameters μ_Ω and ϑ^2 of the given ensemble of specimens are evaluated under a constant-amplitude load.

Using Eq. (15), the realization $\Omega(\zeta*, \Delta S)$ in Eq. (17) is obtained as a deterministic function of ΔS by having $\xi(\zeta*)$ known individually for each of the 35 specimens. The duration of overload effect is different in each specimen and is directly controlled by $\eta(\zeta*)$, which is identified by fitting the crack growth profile for the duration of overload effect in each sample path.

Goodness of the hypothesis that the random variable $\Omega(\zeta,\Delta S)$ for a fixed ΔS, is two-parameter ($r=2$) log–normal distributed is tested by both χ^2 and Kolmogorov–Smirnov tests. The data set is partitioned into $L=6$ segments. With $(L-r-1)=3$ degrees of freedom, the χ^2 test shows that the hypothesis of two-parameter log–normal distribution of Ω passes the 30% significance level. The hypothesis of two-parameter log–normal distribution of Ω also passes the 20% significance level of the Kolmogorov–Smirnov test. The analytically derived log–normal distributed pdf of $\Omega(\zeta,\Delta S)$ closely agrees with the corresponding histograms generated from experimental data based on the fixed $\Delta S=42.2$ MPa which is the effective stress range for an equivalent center-cracked Ti–6Al–4 V alloy specimen under constant-amplitude load. The effects of second-order statistics of the (relatively small) noise $\rho(\zeta,t)$ are neglected similar to those for 2024-T3 and 7075-T6 alloys reported earlier under constant-amplitude load.

The random parameter $\eta(\zeta)$ in Eq. (3) which controls the overload effect duration must be positive (with probability 1) to assure stability of the crack opening stress S^o in Eq. (2). Accordingly, we hypothesize several distributions for $\eta(\zeta)$, including uniform distribution ($r=2$) and log–normal distribution ($r=2$). With a six segment histogram of $\eta(\zeta)$, the χ^2-test and the Kolmogorov–Smirnov test hypotheses pass the 5% significance level. Although neither of these two pdfs matches very closely with the histograms of the experimental data, the uniform distribution is apparently a more viable option. The conclusion is that the pdf of the random parameter $\eta(\zeta)$ is not yet firmly established due to the small sample size (35 in this case) of the test data.

The log–normal distributions of $\Omega(\zeta,\Delta S)$ and $\eta(\zeta)$ are simulated by taking exponential of the normal random number generator and the uniform distribution of $\eta(\zeta)$ by the uniform random number generator. Upon identification of the statistics of $\Omega(\zeta,\Delta S)$ and $\eta(\zeta)$, the whole data set is regenerated by Monte Carlo simulation of the proposed stochastic model; the pertinent model parameters are listed in Table 2. Fig. 2 shows a comparison of the random fatigue test data of Ti–6Al–4 V alloy under overload in the top plate with the corresponding model prediction in the bottom plate. The log–normal distribution of $\Omega(\zeta,\Delta S)$ is simulated by taking

Table 2
Model parameters for Ti–6Al–4 V alloy based on Ghonem and Zeng data set

Data set and material type	Effective stress range $\Delta S \equiv S^{\max} - S^{\mathrm{o}}$ (MPa)	m (dimensionless)	μ_Ω (SI units)	$(\Delta S)^m \sigma_\Omega / \mu_\Omega$ (SI units)	μ_ρ (dimensionless)	$(\Delta S)^m \sigma_\rho / \mu_\rho$ (SI units)
Ghonem and Zeng Ti–6Al–4 V	41.8	2.93	4.636×10^{-10}	8.496×10^{3}	1.0	0

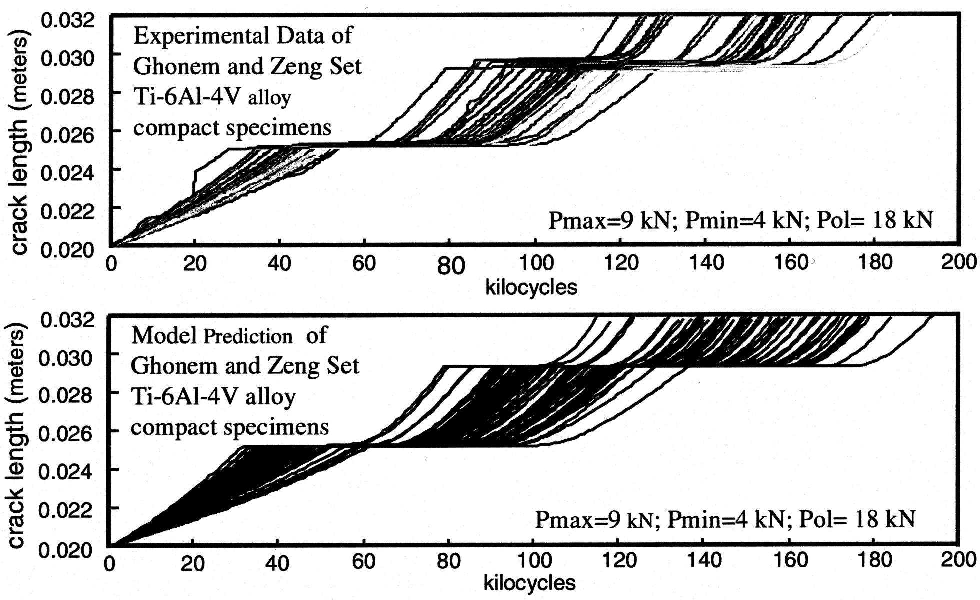

Fig. 2. Model verification with experimental data for compact specimens of Ti–6Al–4 V alloy.

exponential of the standard normal random number generator and the uniform distribution of $\eta(\zeta)$ by the uniform random number generator. The model also matches the ensemble of experimental data for all individual specimens. As an example of how the model fits individual

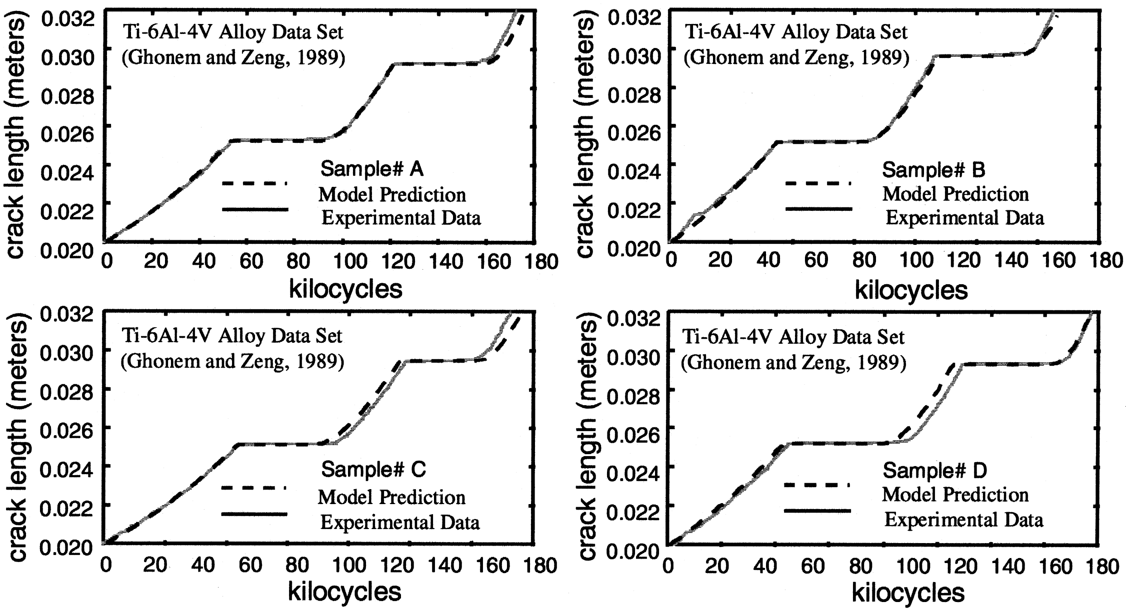

Fig. 3. Model verification with experimental data of individual specimens of Ti–6Al–4 V alloy.

samples, Fig. 3 compares the model predictions with experimental data for four typical individual samples of Ti–6Al–4 V compact specimens [5].

5. Summary and conclusions

This paper presents a stochastic model of fatigue crack propagation for risk analysis and life prediction of mechanical structures and machine components in complex systems (e.g. aircraft, spacecraft, ships and submarines, and power plants). The stochastic model is built upon a fracture-mechanics-based state-space model of fatigue crack growth where the crack length and crack opening stress are the state variables. The deterministic state-space model of fatigue crack growth captures the essential physical phenomena (e.g. crack retardation and sequence effects) that occur under variable-amplitude loading. The details of model development and validation by comparison with experimental data are reported by [11]. The stochastic model is primarily applicable to fatigue crack growth in the Paris regime where the stress intensity factor range ΔKcan be separated as the product of functions of and the effective stress range and crack length, $c(\zeta,t)$. For near-threshold and high-ΔKregimes, the exponent parameter, m, used in the stochastic model may need to be modified.

For constant-amplitude loading, the uncertainties of crack propagation accrue primarily from a single log–normal distributed random parameter $\Omega(\zeta,\Delta S)$ associated with individual specimens and ΔS and, to a much lesser extent, from the (cycle-dependent) random noise $\rho(\zeta,t)$ due to material inhomogeneity. For life prediction and risk analysis, the random process $\rho(\zeta,t)$ can be set to unity (i.e. its expected value) with no significant loss of accuracy. For variable-amplitude loading, the uncertainties in crack propagation are affected by one additional random parameter $\eta(\zeta)$ that controls the duration of the overload effect. This random parameter is identified as a constant at different crack lengths and different load spectra for a given specimen. Based on the available fatigue test data of 2024-T3, 7075-T6, and Ti–6Al–4 V, the following three conjectures (that need to be verified with additional experimental data) are proposed for center-cracked (or equivalent) specimens under both constant-amplitude and variable-amplitude loading:

- For a given material, μ_Ω is a (stress-independent) constant and σ_Ω^2 is proportional to $(\Delta S)^{-2\ m}$;
- The random process Ω(Δi S) for a fixed ΔS can be approximated as two-parameter log–normal distributed.
- The random variable $\eta(\zeta)$ can be approximated as uniformly distributed.

Each individual sample path under variable-amplitude loading can be statistically reproduced for a given ΔS from the two random parameters, $\Omega(\zeta,\Delta S)$ and $\eta(\zeta)$, as discussed above. Using this procedure, the proposed stochastic model for fatigue crack propagation has been verified with experimental data for: (i) 2024-T3 and 7075-T6 aluminum alloys at different levels of constant-amplitude loading; and (ii) Ti–6Al–4 V alloy under variable-amplitude loading comprising constant-amplitude loading coupled with overloads at different cycles.

Potential applications of the stochastic crack propagation model include: (i) remaining life prediction of machinery components; (ii) generation of alerts and warnings for operational support and safety enhancement; (iii) equipment readiness assessment and failure prognosis based on current condition and projected usage; and (iv) formulation of decision policies for maintenance scheduling in real time based on the up-to-date information of machinery operation history and anticipated usage.

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