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Compensatability and optimal compensation under randomly varying distributed delays

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This paper establishes necessary and sufficient conditions for existence, uniqueness and global optimality of the Linear Quadratic Coupled Delay Compensator (LQCDC) which is designed to circumvent the detrimental effects of randomly varying delays from sensor to controller and from controller to actuator as well as the time skew caused by mis-synchronization of sensor and controller sampling instants. These conditions are derived on the basis of the concepts of stabilizability, detectability and compensatability in the mean square sense. In the absence of random delays from sensor to controller and from controller to actuator, it has been shown that LQCDC problems reduce to the classical Linear Quadratic Gaussian (LQG).

1. Introduction

Randomly varying distributed delays, induced by a computer communication network, may degrade stability and performance of real-time systems because timely transfer of sensor and control signals from one device to another is not guaranteed (Ray and Halevi 1988, Nilsson 1998). Delay problems, in general, have been investigated by many researchers. For example, Bernstein *et al.* (1986) address the issues of data processing delays in the computation of control laws where the command, computed in the previous sampling period, is applied during the current sampling period.

An output feedback controller, called the Linear Quadratic Random Coupled Compensator (LQCDC), has been proposed by Tsai and Ray (1997) to circumvent the detrimental effects of the randomly varying distributed delays from sensor to controller and from controller to actuator. The concept of composite design of the stochastic controller and state estimator is brought into the LQCDC to resolve the problem of violation of the certainty equivalence principle. Two pairs of modified matrix Riccati and matrix Lyapunov equations for LQCDC are coupled by a projection matrix whose column and row spaces are respectively the control and estimation subspaces. Due to the coupling of control and estimation in the presence of randomly varying distributed delays, the essential role that projection matrix and its factorization play is investigated. Unlike full-order or reduced-order output feedback compensation (Kwakernaak and Sivan 1972), the LQCDC has a full-order state estimator and a state-augmented controller.

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De Koning (1992) introduces the concept of compensability for analysis of dynamic systems with multiplicative white noise where optimality and stabilizability under random delays are not considered. The present paper is an extension of our earlier work (Tsai and Ray 1997) and addresses the issue of optimal compensation of the LQCDC in the mean square (ms) sense. A set of ms compensatability conditions (for the closed-loop control system) and ms detectability conditions (for the models of performance cost evaluation and noise covariance) is shown to be necessary and sufficient, in general, for the existence of a unique optimal LQCDC. The conditions reduce to those for stabilizability and detectability in the standard LQG setting in the absence of random delays, both from sensor to controller and from controller to actuator, and the time skew, caused by mis-synchronization of sensor and controller sampling instants.

2. System model and the LQRDC law

Let the plant dynamics and disturbances, and the (delayed and noisy) sensor data be modelled as

$$\xi(t) = a(t)\xi(t) + b(t)u(t) + g(t)\omega(t) \tag{1}$$

$$y(t) = c(t)\xi(t) + \upsilon(t) \tag{2}$$

where the plant state vector $\xi(t) \in \Re^n$, the control vector $u(t) \in \Re^m$, the measurement vector $y(t) \in \Re^r$, the real deterministic matrices $a(t)$, $b(t)$ and $g(t)$, and the stochastic matrix $c(t)$ are of compatible dimensions, the vectors $\omega(t)$ and $\upsilon(t)$ represent zero-mean, mutually independent and strictly stationary white noise for the plant and sensor models, with covariance matrices, $V_1 \geq 0$ and $V_2 > 0$, respectively. For discrete-time control, let the sensor and controller have an identical sampling period T although their sampling instants may not be synchronized. The time difference between sensor and controller sampling instants is called the time skew, Δ, which is bounded by the sampling period T. In order to distinguish the *sensor time frame* from the *controller time*

http://www.tandf.co.uk/JNLS/con.htm
http://www.taylorandfrancis.com/JNLS/con.htm

frame, the superscript 's' is used to indicate signals based on *the sensor time frame*, i.e. $(\bullet)^{s}_{k} = (\bullet)_{k-\delta}$ where $(\bullet)$ is any arbitrary vector or matrix; the normalized time skew δ in the subscript is defined as $\delta \equiv \Delta / T$. Unless otherwise specified, the discrete-time frame k is based on the controller time frame instead of the sensor time frame.

Following Ray *et al.* (1993) and Tsai and Ray (1997), the sensor data for generating the $(k+1)$th control command, denoted as z_k, is subjected to binary random delays such that either $z_k = y^{s}_{k}$ or $z_k = y^{s}_{k-1}$, depending on whether the fresh sensor data or the previous data is to be used for generation of u_{k+1}. That is

$$z_k = (1 - \varsigma_k) y^{s}_{k} + \varsigma_k y^{s}_{k-1} \tag{3}$$

where the white delay sequence $\{\varsigma_k\}$ from sensor to controller, which is independent of disturbances $\{\omega^{s}_{k}\}$ and sensor noise $\{\upsilon^{s}_{k}\}$, has a binary distribution having the expectation

$$E[\varsigma_k] = 1 - \alpha_k \quad \text{and} \quad E[(\varsigma_k)^2] = 1 - \alpha_k \tag{4}$$

where α_k is the probability of timely arrival of sensor output y^{s}_{k} at the controller.

With the augmentation of plant state to include state estimate and the three past consecutive control commands, the closed-loop system under randomly varying distributed delays is therefore established as follows.

Closed-loop model:

$$\tilde{x}_{k+1} = \tilde{A}_k \tilde{x}_k + \tilde{G}_k \Omega_k \tag{5}$$

where

$$\tilde{x}_k = [(\xi^{s}_{k})^{\mathrm{T}} u^{\mathrm{T}}_{k-1} u^{\mathrm{T}}_{k-2} u^{\mathrm{T}}_{k-3} (\hat{\xi}^{s}_{k})^{\mathrm{T}}] \tag{6a}$$

$$\tilde{A}_k = \begin{bmatrix} A^{k}_{\mathrm{au}} + \Delta^{k}_{A} & B^{k}_{\mathrm{au}} \hat{F}_k \\ \hat{K}_k C^{k}_{\mathrm{au}} + \Delta^{k}_{C} & \tilde{L}_{c\ell} \end{bmatrix} \tag{6b}$$

$$\hat{L}_{c\ell} = L_k + \Delta^{k}_{0} \tag{7a}$$

$$\Delta^{k}_{0} = \beta^{k}_{0} + \hat{F}_k \tag{7b}$$

$$\Delta^{k}_{A} = B^{k}_{\mathrm{au}} \Pi_{01} \tag{8a}$$

$$\Pi_{01} = [0_{3m\times n} \quad I_{3m}] \tag{8b}$$

$$\hat{f}^{k}_{\mathrm{au}} = [\hat{f}^{k}_{1} \quad \hat{f}^{k}_{2} \quad \hat{f}^{k}_{3}] \tag{8c}$$

$$\Delta^{k}_{C} = [0_n \quad \Delta^{k}_{q1} \quad \Delta^{k}_{q2} \quad \Delta^{k}_{q3}] \tag{9a}$$

$$\Delta^{k}_{q1} = \beta^{k}_{1} + \beta^{k}_{1} \hat{f}^{k}_{1} \tag{9b}$$

$$\Delta^{k}_{q2} = \beta^{k}_{2} + \beta^{k}_{0} \hat{f}^{k}_{2} \tag{9c}$$

$$\Delta^{k}_{q3} = \beta^{k}_{0} \hat{f}^{k}_{3} \tag{9d}$$

$$\tilde{G}_k = \begin{bmatrix} I_{n\times n} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & g^{k}_{q1} & g^{k}_{q2} & g^{k}_{q3} \end{bmatrix} \tag{10a}$$

$$g^{k}_{q1} = -\varsigma_k \hat{K}_k c^{s}_{k-1} (\Phi^{s}_{k-1})^{-1} \tag{10b}$$

$$g^{k}_{q2} = (1 - \varsigma_k) \hat{K}_k \tag{10c}$$

$$g^{k}_{q3} = \varsigma_k \hat{K}_k \tag{10d}$$

$$\tilde{\Omega}_k = [(\omega^{s}_{k})^{\mathrm{T}} \quad (\omega^{s}_{k-1})^{\mathrm{T}} \quad (\upsilon^{s}_{k})^{\mathrm{T}} \quad (\upsilon^{s}_{k-1})^{\mathrm{T}}] \tag{11}$$

Remark 1: if $\Delta^{k}_{A} = 0$ and $\Delta^{k}_{C} = 0$, the closed-loop system matrix, $\tilde{A}_k$, becomes similar to that in the Linear Quadratic Guassian (LQG) model.

Tsai and Ray (1997) have constructed the *linear quadratic coupled delay compensation* (LQCDC) law, characterized by $\{\hat{L}^{*}_{k}, \hat{K}^{*}_{k}, (\hat{F}^{k}_{\mathrm{au}})^{*}\}$, in order to minimize the following performance cost functional

$$\tilde{J}_k(\hat{L}^{+}_{k}, \hat{K}^{+}_{k}, \hat{F}^{k}_{\mathrm{au}}, \tilde{Z}_{k-1}) = \tfrac{1}{2} E[\tilde{x}^{\mathrm{T}}_{k} \tilde{R}_k \tilde{x}_k \mid \tilde{Z}_{k-1}] + E[\tilde{J}^{*}_{k+1}(\tilde{Z}_k) \mid \tilde{Z}_{k-1}] \tag{12a}$$

$$\tilde{J}_k(\tilde{Z}_{k-1}) = \tilde{J}_k(\hat{L}^{*}_{k}, \hat{K}^{*}_{k}, (\hat{F}^{k}_{\mathrm{au}})^{*}, \tilde{Z}_{k-1}) \tag{12b}$$

$$\tilde{J}^{*}_{N}(\tilde{Z}_{N-1}) = \tfrac{1}{2} E[\tilde{x}^{\mathrm{T}}_{N} \tilde{R}_k \tilde{x}_N \mid \tilde{Z}_{N-1}] \tag{12c}$$

$$\tilde{R}_k = \begin{bmatrix} R_1 & 0_{n\times(n+3m)} \\ 0_{(n+3m)\times n} & (F^{k}_{\mathrm{au}})^{\mathrm{T}} R_1 F^{k}_{\mathrm{au}} \end{bmatrix} \tag{12d}$$

$$R_N = \begin{bmatrix} \bar{\bar{R}} & 0_{n\times(n+3m)} \\ 0_{(n+3m)\times n} & 0_{(n+3m)\times(n+3m)} \end{bmatrix} \tag{12e}$$

The superscript '*' is used to denote that optimality is achieved. $\tilde{Z}_k \equiv \{\varsigma_k, \varsigma_{k-1}, \ldots, \varsigma_1; t^{k}, t^{k-1}, \ldots, t^{1}\}$ is the random delay history; N is the time horizon over which the performance is evaluated; $R_1 \geq 0$ and $R_2 > 0$ are the state deviation and control penalty matrices, respectively, for $k < N$; and $\bar{\bar{R}} \geq 0$ is the state deviation matrix at $k = N$.

3. Compensatability and adjoint systems

Let B^{s}_{k} be the discretized input matrix corresponding to the whole sensor sampling period. That is

$$B_k^{\mathrm{s}} = \int_{(k-\delta)T}^{(k+1-\delta)T} \mathrm{d}\tau\, \Phi((k+1-\delta)T, \tau) b(\tau) \quad (13)$$

where $\Phi(\bullet, \bullet)$ and $b(\bullet)$ are the continuous-time transition matrix and the input matrix, receptively.

Definition 1: In the absence of plant disturbance and sensor noise

$$\tilde{X}_{k+1} = \tilde{A}_k \tilde{X}_k \quad \text{for } k = 0, 1, 2, 3, \ldots \quad (14)$$

where $\tilde{X}_k \in R^{2n+3m}$ is the augmented state vector and $\tilde{A}_k$ is the stochastic closed-loop system matrix for the system in (14), characterized by $(\Phi_k^{\mathrm{s}}, B_k^{\mathrm{s}}, c_k^{\mathrm{s}}, \Delta, t^k, \varsigma_k)$, which is called *ms compensatable* if there exists a compensator triplet $(\hat{L}_k, \hat{K}_k, \hat{F}_{\mathrm{au}}^k)$ such that $\tilde{A}_k$ is ms stable.

The following properties of ms compensatability are stated below:

(1) If Φ_k^{s} is stable, $(\Phi_k^{\mathrm{s}}, B_k^{\mathrm{s}}, c_k^{\mathrm{s}}, \Delta, t^k, \varsigma_k)$ is ms compensatable.
(2) Both $(\Phi_k^{\mathrm{s}}, B_k^{\mathrm{s}})$ and $((\Phi_k^{\mathrm{s}})^{\mathrm{T}}, (c_k^{\mathrm{s}})^{\mathrm{T}})$ are stabilizable iff $(\Phi_k^{\mathrm{s}}, B_k^{\mathrm{s}}, c_k^{\mathrm{s}}, 0, 0, 0)$ is *ms* compensatable $\forall k$.

Property (1) is evident because $\hat{L}_k = 0$, $\hat{K}_k = 0$ and $\hat{F}_{\mathrm{au}}^k = 0$ is a solution for the regulator problem, i.e. the closed-loop system remains stable under zero input. The second property of LQCDC, as a matter of fact, reduces to typical characteristics of LQG. In other words, stabilizability of the state feedback system and its adjoint ensures compensatability of the closed-loop system and vice versa. Equivalently, in LQG, the concept of compensatability in the usual sense reduces to stabilizability and detectabilty. However, in LQCDC, mean square stabilizability and detectabilty jointly do not imply mean square compensatabilty because of breakdown of the separation principle. Conversely, it is shown that mean square compensatability indeed implies mean square stabilizability of LQCDC and its adjoint. Therefore, mean square compensatability is a stronger condition than the combination of mean square stabilizability and detectability.

This concept motivates the construction of the adjoint of LQCDC so that the important features of ms compensatability can be revealed. For the adjoint system

$$\tilde{X}_{k-1}^{\mathrm{ad}} = \tilde{A}_k^{\mathrm{T}} \tilde{X}_k^{\mathrm{ad}} + \tilde{G}_k^{\mathrm{ad}} \tilde{\Omega}_k^{\mathrm{ad}} \quad \text{for } k = N-1, N-2, \ldots \quad (15)$$

where the state vector $\tilde{X}_k^{\mathrm{ad}} \in \mathcal{R}^{2n+3m}$; the stochastic matrix $\tilde{A}_k$ is as defined in (6 *b*), and the system noise vector, $\tilde{\Omega}_k^{\mathrm{ad}}$, and its input matrix, $\tilde{G}_k^{\mathrm{ad}}$, are defined as follows

$$\tilde{\Omega}_k^{\mathrm{ad}} = \begin{bmatrix} \omega_k^{\mathrm{ad}} \\ \upsilon_k^{\mathrm{ad}} \end{bmatrix}; \quad \text{and } \tilde{G}_k^{\mathrm{ad}} = \begin{bmatrix} I_{n\times n} & 0 & 0 & 0 & 0 \\ 0 & \hat{f}_1^k & \hat{f}_2^k & \hat{f}_3^k & \hat{F}_k \end{bmatrix}^{\mathrm{T}} \quad (16)$$

where zero-mean ω_k^{ad} and υ_k^{ad} are mutually independent with covariance matrices R_1 and R_2 respectively. The final state $\tilde{X}_{N-1}^{\mathrm{ad}}$ is zero-mean with covariance $\tilde{R}$. The system (15) has the following conditional covariance dynamics as a regulation problem

$$P_{k-1} = E[\tilde{A}_k^{\mathrm{T}} P_k \tilde{A}_k \mid \tilde{Z}_k] + \tilde{R}_k \quad \text{for } 1 \le k \le N-1 \quad (17a)$$

where conditional covariance

$$P_k \equiv E[\tilde{X}_k^{\mathrm{ad}} (\tilde{X}_k^{\mathrm{ad}})^{\mathrm{T}} \mid \tilde{Z}_{k+1}]$$

with terminal condition $P_{N-1} = \tilde{R}$. In contrast, the original conditional covariance dynamics of LQCDC (equation (30) in Tsai and Ray 1997) is described as

$$Q_{k+1} = E[\tilde{A}_k Q_k \tilde{A}_k^{\mathrm{T}} \mid \tilde{Z}_k] + \tilde{V}_{\mathrm{eq}}^k \quad \text{for } 0 \le k \le N-1 \quad (17b)$$

$$\tilde{V}_{\mathrm{eq}}^k = \mathrm{diag}\,[V_1 \quad 0 \quad 0 \quad 0 \quad \tilde{V}_2^k] \quad (17c)$$

$$\tilde{V}_2^k = \hat{K}_k (V_2 + E[\varsigma_k^2 \mid \tilde{Z}_k] c_{k-1}^{\mathrm{s}} (\Phi_{k-1}^{\mathrm{s}})^{-1} \times V_1 [c_{k-1}^{\mathrm{s}} (\Phi_{k-1}^{\mathrm{s}})^{-1}]^{\mathrm{T}}) K_k^{\mathrm{T}} \quad (17d)$$

with a given initial condition for Q_0. Since (17 *a*) can also be directly constructed by interchanging the roles of system noise covariance and state deviation penalty matrix under the closed loop of LQCDC, the system in (15) is hereby defined and named as the adjoint of LQCDC with replacement of the system matrix by its transpose and reversal of the time frame in the backward direction. It is noted that the conditional covariance P_k of the adjoint is also the adjoint of the conditional covariance Q_k of the origianl system. That is, based on the adjoint system in (15), the same LQCDC law can be induced via exchange of the roles of Q_k and P_k. On the basis of the definitions of ms compensatabilty and adjoint system, the next section is devoted to development of the conditions for the existence and uniqueness of LQCDC. To conclude this section, a definition is stated that will be used later.

Definition 2: (A_k, B_k, C_k) is called *ms detectable* if (A_k, C_k) and $(A_k^{\mathrm{T}}, B_k^{\mathrm{T}})$ are both *ms detectable.*

4. Necessary and sufficient conditions for optimality of LQCDC

The control and estimation laws and the two pairs of modified matrix Riccati and Lyapunov equations exactly constitute the necessary optimality conditions of LQCDC (Tsai and Ray 1997). Therefore, this section will be mainly devoted to the establishment of sufficient optimality conditions. As defined earlier, a compensator is called ms stabilizing if the mean square value of the state vector converges to zero as time runs to infinity, i.e. $E\left[\|\tilde{X}_k\|^2\right] \to 0$ as $k \to \infty$. The incremental cost functional, $J_k \equiv \frac{1}{2} E\left[\tilde{X}_k^{\mathrm{T}} \tilde{R}_k \tilde{X}_k \mid \tilde{Z}_{k-1}\right]$, will be shown to be independent of the initial values of plant state and its estimate as the steady state of LQCDC is reached. Let the closed-loop system matrix of LQCDC $(\tilde{A}_k)$ be ms stable. Then, the compensator $(\hat{L}_k, \hat{K}_k, \hat{F}^k_{\mathrm{au}})$ is ms stabilizing and, therefore, from the equation set (17), the steady-state expected values of conditional covariance matrices, $\bar{Q}_{\mathrm{s}} \equiv E\left[\lim_{k\to\infty} Q_k\right]$ and $\bar{P}_{\mathrm{s}} \equiv E\left[\lim_{k\to\infty} P_k\right]$, both exist. Their uniqueness will be discussed later.

Remark 2: If $\tau_k = I_{n+3m}$, then the modified matrix Riccati and Lyapunov equations are separable, i.e. the projection matrix becomes trivial. In other words, if the state estimator is augmented from dimension n to dimension $n + 3m$, the property of certainty equivalence holds. However, it is redundant to estimate the exactly known values of the three consecutive past control commands, namely $\{u_{k-1}, u_{k-2}, u_{k-3}\}$, which are already stored in the control buffer at time k (Liou and Ray 1991, Ray 1994). On the other hand, if the states of the estimator are augmented to include the three consecutive past control commands, the closed-loop model becomes non-minimal since the three consecutive past control commands are twice accounted for. It is inappropriate to let $\tau_k = I_{n+3m}$ and hence the projection matrix may not be identity in the LQCDC problem. It is not necessary to be symmetric, either.

Lemma 1: *If $R_1 > 0$ or $(\Phi^{\mathrm{s}}_k, R_1^{1/2})$ is ms detectable, then $(\tilde{A}_k, \tilde{R}_k^{1/2})$ is ms detectable.*

Proof: If $R_1 > 0$, there exists a $\gamma \in (0, \infty)$ such that $\gamma \tilde{R}_k \geq I_{2n+3m}$. Hence, $E\left[\tilde{x}_k^{\mathrm{T}} \tilde{x}_k\right] \leq \gamma E\left[\tilde{x}_k^{\mathrm{T}} \tilde{R}_k \tilde{x}_k\right] \forall k$. Therefore, $E\left[\tilde{x}_k^{\mathrm{T}} \tilde{R}_k \tilde{x}_k\right] = 0 \ \forall k \Rightarrow E\left[\tilde{x}_k^{\mathrm{T}} \tilde{x}_k\right] = 0$ as $k \to \infty$. That is, $(\tilde{A}_k, \tilde{R}_k^{1/2})$ is ms detectable.

Next we prove that $(\Phi^{\mathrm{s}}_k, R_1^{1/2})$ ms detectable $\Rightarrow (\tilde{A}_k, \tilde{R}_k^{1/2})$ ms detectable. Let $E\left[\tilde{x}_k^{\mathrm{T}} \tilde{R}_k \tilde{x}_k\right] = 0 \ \forall k$; then

$$E\left[\tilde{x}_k^{\mathrm{T}} \tilde{R}_k \tilde{x}_k\right] = E\left[(\xi^{\mathrm{s}}_k)^{\mathrm{T}} R_1 \xi^{\mathrm{s}}_k\right] + E\left[u_k^{\mathrm{T}} R_2 u_k\right] = 0 \ \forall k$$

which implies $E\left[u_k^{\mathrm{T}} R_2 u_k\right] = 0 \ \forall k$. Since $R_2 > 0$, then $u_k = 0 \ \forall k$ almost surely. Under the LQCDC law (Tsai and Ray 1997), $u_k = \hat{F}_k \hat{\xi}^{\mathrm{s}}_k + \hat{f}^k_1 u_{k-1} + + \hat{f}^k_2 u_{k-2} + \hat{f}^k_3 u_{k-3}$. Therefore,

$$u_k = 0 \ \forall k \text{ almost surely} \Rightarrow \hat{\xi}^{\mathrm{s}}_k = 0 \ \forall k \text{ almost surely}$$

for any choice of $u_k = \hat{F}_k \hat{\xi}^{\mathrm{s}}_k + \hat{f}^k_1 u_{k-1} + + \hat{f}^k_1 u_{k-2} + \hat{f}^k_3 u_{k-3}$. From definition of ms detectability of $(\Phi^{\mathrm{s}}_k, R_1^{1/2})$,

$$E\left[(\xi^{\mathrm{s}}_k)^{\mathrm{T}} R_1 \xi^{\mathrm{s}}_k\right] = 0 \ \forall k \Rightarrow E\left[(\xi^{\mathrm{s}}_k)^{\mathrm{T}} \xi^{\mathrm{s}}_k\right] \to 0 \text{ as } k \to \infty$$

Since $u_k = 0$, $\hat{\xi}^{\mathrm{s}}_k = 0 \ \forall k$ almost surely

$$E\left[\tilde{x}_k^{\mathrm{T}} \tilde{R}_k \tilde{x}_k\right] = 0 \ \forall k \Rightarrow E\left[\tilde{x}_k^{\mathrm{T}} \tilde{x}_k\right] \to 0 \text{ as } k \to \infty$$

That is, $(\tilde{A}_k, \tilde{R}_k^{1/2})$ is ms detectable. □

Corollary to Lemma 1: *If $V_1 > 0$ or $((\Phi^{\mathrm{s}}_k)^{\mathrm{T}}, V_1^{1/2})$ is ms detectable, then $(\tilde{A}_k^{\mathrm{T}}, (\tilde{V}^k_{\mathrm{eq}})^{1/2})$ is ms detectable.*

Proof: The proof is based on the duality of the results in Lemma 1 (see Tsai (1995) for details). □

Remark 3: In the LQR problem, detectability of the system matrix and a square root of the state penalty matrix ensure closed stability. On the basis of this concept, Lemma 1 is applied to the LQCDC system to ensure ms stability. However, unlike LQG, since the LQCDC system is not decoupled for control and estimation, the concepts of conventional stabilizability and detectability in the usual sense are extended to mean square detectability in the triplet (A_k, B_k, C_k), so that the convergence and mean square compensatabilty of LQCDC are guaranteed.

Up to now, the LQCDC has been described as time varying. As the recursive relations in (17*a*) and (17*b*) proceed, the solutions of the modified matrix Riccati and Lyapunov equations exist and are unique in a finite-time horizon. However, as time goes to infinity, the recursions may not be unconditionally convergent, and the problem of existence of an LQCDC solution arises. Moreover, uniqueness of the LQCDC solution is not assured even if convergence is reached. In other words, the steady-state solution of LQCDC needs to be specified. Especially in numerical analysis and implementation, the steady-state compensator, instead of the time-dependent compensator, is usually employed. Nevertheless, in the following proposition it will be proved that ms stability and quadratic optimality of time-varying LQCDC can also be guaranteed provided that the randomly delayed system, $(\Phi^{\mathrm{s}}_k, B^{\mathrm{s}}_k, c^{\mathrm{s}}_k, \Delta, t^k, \varsigma_k)$, is ms compensatable and the triplet $\{\Phi^{\mathrm{s}}_k, V_1^{1/2}, R_1^{1/2}\}$ is ms detectable, or simply $\{R_1 > 0, V_1 > 0\}$, where $\Phi^{\mathrm{s}}_k, B^{\mathrm{s}}_k$ and c^{s}_k denote the system, input and output matrices of the plant respectively; Δ, t^k and ς_k are the time skew, input delay sequence and output delay sequence in order while R_1 and V_1 are the state deviation and plant disturbance matrices respectively.

Proposition 1: *For a fixed projective factorization, $\tau_k = G^{\mathrm{T}}_{k+1} \Gamma_k$, a unique steady state-linear quadratic coupled delay compensator (LQCDC) exists if the random delay compensated system, characterized by $(\Phi^{\mathrm{s}}_k, B^{\mathrm{s}}_k, c^{\mathrm{s}}_k, \Delta, t^k, \varsigma_k)$, is ms compensatable and either the triplet $\{\Phi^{\mathrm{s}}_k, V_1^{1/2}, R_1^{1/2}\}$ is ms detectable or the state deviation matrix and plant disturbance covaraince are both positive definite, $\{R_1 > 0, V_1 > 0\}$.*

Proof: Since $(\Phi_k^s, B_k^s, c_k^s, \Delta, t^k, \varsigma_k)$ is mean square compensatable, there exists a compensator triplet $(\hat{L}_k, \hat{K}_k, \hat{F}_{au}^k)$ such that the closed-loop system matrix $\tilde{A}_k$ is ms stable $\forall k$. Hence, for such a compensator triple $(\hat{L}_k, \hat{K}_k, \hat{F}_{au}^k)$, the pair of necessary optimality conditions (17a) and (17b) at least have a non-negative definite solution, (Q_k, P_k) $\forall k$, because (Q_k, P_k) $\forall k$ always exists and $(\tilde{A}_k^{\mathrm{T}}, (\tilde{V}_{eq}^k)^{1/2}, \tilde{R}_k^{1/2})$ is ms detectable from Lemma 1 and its corollary; $\tilde{A}_k$ is ms stable $\forall k$. That is, the solutions calculated from the necessary optimality conditions ensure the existence of LQCDC such that the closed-loop system is stable in the mean square sense. This concludes that a stabilizing LQCDC, restricted to the set of stable minimum compensator, exists. Therefore, the rest of this proof is devoted to uniqueness of LQCDC.

Define the operator, $\boldsymbol{R}_{\tilde{A}}^k : \Re^{2n+3m} \to \Re^{2n+3m}$ as the conditional Riccati transformation, i.e. $\boldsymbol{R}_{\tilde{A}}^k X = E[\tilde{A}_k X \tilde{A}_k^{\mathrm{T}} | \tilde{Z}_k]$ for any square matrix $X \in \Re^{2n+3m}$. Since $(\Phi_k^s, B_k^s, c_k^s, \Delta, t^k, \varsigma_k)$ is ms compensatable and $\tilde{A}_k$ has been proved ms stable above, under the LQCDC law, $Q_s \equiv \lim_{k\to\infty} Q_k$ and $P_s \equiv \lim_{k\to\infty} P_k$ are existent and unique. They can be presented as follows (Chen 1984)

$$Q_S = \sum_{k=0}^{\infty} \prod_{j=0}^{k} \boldsymbol{R}_{\tilde{A}}^k \tilde{V}_{eq}^j \tag{18a}$$

$$P_S = \sum_{k=0}^{\infty} \prod_{j=0}^{k} \boldsymbol{R}_{\tilde{A}^{\mathrm{T}}}^k \tilde{R}_j \tag{18b}$$

where

$$\prod_{j=0}^{k} \boldsymbol{R}_{\tilde{A}}^k \boldsymbol{R}_{\tilde{A}}^k \boldsymbol{R}_{\tilde{A}}^{k-1} \dots \boldsymbol{R}_{\tilde{A}}^1 I_{2n+3m}$$

By partitioning Q_k and P_k into $(n+3m) \times (n+3m)$, $(n+3m) \times n$ and $n \times n$ submatrices respectively, the following two sets of equations hold:

$$\bar{\bar{Q}}_x^k = Q_x^k - \hat{Q}_x^k \tag{19a}$$

$$\hat{Q}_x^k = Q_{xq}^k (Q_q^k)^{-1} (Q_{xq}^k)^{\mathrm{T}} \tag{19b}$$

$$\bar{\bar{P}}_x^k = P_x^k - \hat{P}_x^k \tag{20a}$$

$$\hat{P}_x^k = P_{xq}^k (P_q^k)^{-1} (P_{xq}^k)^{\mathrm{T}} \tag{20b}$$

where

$$Q_k = \begin{bmatrix} Q_x^k & Q_{xq}^k \\ (Q_{xq}^k)^{\mathrm{T}} & Q_q^k \end{bmatrix} \tag{21a}$$

$$P_k = \begin{bmatrix} P_x^k & P_{xq}^k \\ (P_{xq}^k)^{\mathrm{T}} & P_q^k \end{bmatrix} \tag{21b}$$

$$\hat{Q}_x^{k+1} = \tau_k \hat{\hat{Q}}^{k+1} \tau_k^{\mathrm{T}} \tag{22a}$$

$$\hat{P}_x^k = \tau_k \hat{\hat{P}}_x^k \tau_k^{\mathrm{T}} \tag{22b}$$

Therefore, the limits $\bar{\bar{Q}}_x^s \equiv \lim_{k\to\infty} \bar{\bar{Q}}_x^k$, $\hat{Q}_x^s \equiv \lim_{k\to\infty} \hat{Q}_x^k$, $\bar{\bar{P}}_x^s \equiv \lim_{k\to\infty} \bar{\bar{P}}_x^k$ and $\hat{P}_x^s \equiv \lim_{k\to\infty} \hat{P}_x^s \equiv \lim_{k\to\infty} \hat{P}_x^k$ are all unique, for (Q_s, P_s) is unique. So is the steady-state projection matrix, $\tau_s \equiv \lim_{k\to\infty} \tau_k$, for $\tau_k = \hat{Q}_x^{k+1} \hat{P}_x^k (\hat{Q}_x^{k+1} \hat{P}_x^k)^{\#}$ where '#' indicates general group matrix inverse. From De Koning (1992), the conditional Riccati transformations of the modified matrix Riccati and Lyapunov equations in LQCDC, characterized by $[A_{au}^k + \Delta_A^k - \hat{Q}_s^k (\hat{V}_{2s}^k)^{-1} C_{au}^k - G_{k+1}^{\mathrm{T}} \Delta_C^k]$ and $[A_{au}^k - B_{au}^k (\hat{R}_{2s}^k)^{-1} \hat{P}_{xs}^k]$, are both ms stable since $\tilde{A}_k$ is ms stable. For fixed τ_s and $(\bar{\bar{Q}}_x^s, \bar{\bar{P}}_x^s)$, and ms stability of $[A_{au}^k + \Delta_A^k - \hat{Q}_s^k (\hat{V}_{2s}^k)^{-1} C_{au}^k - C_{au}^k - G_{k+1}^{\mathrm{T}} \Delta_C^k]$ and $[A_{au}^k - B_{au}^k (\hat{R}_{2s}^k)^{-1} \hat{P}_{xs}^k]$ for any k, the modified matrix Riccati and Lyapunov equations both converge to unique values, i.e. $\hat{\hat{Q}}_x^s \equiv \lim_{k\to\infty} \hat{\hat{Q}}_x^k$ and $\hat{\hat{P}}_x^s \equiv \lim_{k\to\infty} \hat{\hat{P}}_x^k$ are unique. Since $(\bar{\bar{Q}}_x^s, \hat{\hat{Q}}_x^s, \bar{\bar{P}}_x^s, \hat{\hat{P}}_x^s)$ is unique, so is LQCDC for some fixed projective factorization. □

It follows from Proposition 1 that the condition, either $\{\Phi_k^s, V_1^{1/2}, R_1^{1/2}\}$ is ms detectable or simply $\{R_1 > 0, V_1 > 0\}$, assures the convergence of recursions of $(\bar{\bar{Q}}_x^k, \hat{\hat{Q}}_x^k, \bar{\bar{P}}_x^k, \hat{\hat{P}}_x^k)$ as long as $(\Phi_k^s, B_k^s, c_k^s, \Delta, t^k, \varsigma_k)$ is ms compensatable. Evidently, it is the sufficient optimality condition. On the other hand, if the convergence of recursion $(\bar{\bar{Q}}_x^k, \hat{\hat{Q}}_x^k, \bar{\bar{P}}_x^k, \hat{\hat{P}}_x^k)$ is assured, what condition will make $(\Phi_k^s, B_k^s, c_k^s, \Delta, t^k, \varsigma_k)$ ms compensatable? Since $(\bar{\bar{Q}}_x^k, \hat{\hat{Q}}_x^k, \bar{\bar{P}}_x^k, \hat{\hat{P}}_x^k)$ converges, there exists a pair (Q_S, P_S) which satisfies equations (17a) and (17b) as time goes to infinity. Once (Q_S, P_S) exists, $(\tilde{A}_k)$ is ensured to be mean square compensatable by the condition that either $\{\Phi_k^s, V_1^{1/2}, R_1^{1/2}\}$ is ms detectable or simply $\{R_1 > 0, V_1 > 0\}$, using the same argument as in the proof of Proposition 1. In other words, it is also a necessary condition. This result is summarized in the following proposition.

Proposition 2: *If the triplet* $\{\Phi^s_k, V_1^{1/2}, R_1^{1/2}\}$ *is mean square detectable or* $\{R_1 > 0, V_1 > 0\}$, *the random delay compensated system,* $(\Phi^s_k, B^s_k, c^s_k, \Delta, t^k, \varsigma_k)$, *is mean square compensatable* $\Leftrightarrow (\underline{\bar{\bar{Q}}}^k_x, \underline{\hat{\hat{Q}}}^k_x, \underline{\bar{\bar{P}}}^k_x, \underline{\hat{\hat{P}}}^k_x)$ *converges to unique value.*

Proof: Under the condition given in Proposition 1, it has been proved that $(\Phi^s_k, B^s_k, c^s_k, \Delta, t^k, \varsigma_k)$ is ms compensatable $\Rightarrow (\underline{\bar{\bar{Q}}}^k_x, \underline{\hat{\hat{Q}}}^k_x, \underline{\bar{\bar{P}}}^k_x, \underline{\hat{\hat{P}}}^k_x)$ converges to the unique value. The rationale for the reverse logic is the same as stated above. □

5. Summary and conclusions

This paper establishes necessary and sufficient conditions for existence, uniqueness and global optimality of the Linear Quadratic Coupled Delay Compensator (LQCDC) stochastic optimal control law (Tsai and Ray 1997) based on the concepts of stability, detectability and compensatability in the mean square sense. These conditions also guarantee ms stability of the closed-loop control system. It is shown that, via construction of an adjoint system, the LQCDC law is developed from a pair of dual conditional covariance dynamics. The projection matrix and its factorization play an essential role in the formulation of LQCDC, which dictates the portrayal and connection of control and estimation subspaces. The coupling of control and estimation in the LQCDC problem is once again verified by an oblique projection matrix. The major issues of steady-state properties of LQCDC are summarized below.

(i) Conditions for existence of LQCDC involve the coupling effects of control and estimation due to the presence of induced delays and mis-synchronization between sensor and controller sampling instants. This is in contrast to the stabilizability and detectability conditions of the standard LQG where control and estimation are decoupled. In the formulation of LQCDC, the concept of mean square compensatability is introduced to integrate the properties of stabilizability and detectability in the stochastic sense.

(ii) The pairs of modifed matrix Riccati and Lyapunov equations constitute the necessary conditions for LQCDC. Since the control and estimation are coupled, the column and row spaces of the projection matrix form the subspaces of control and estimation, respectively. These relationships are established through an additional pair of modified matrix Lyapunov equations. For the standard LQG, the projection matrix becomes the identity matrix so that the control and estimation problems become separable. The above necessary conditions are precisely the LQG gain relations, together with a pair of matrix Riccati equations which are used to compute the control and estimation gain matrices.

(iii) Sufficient conditions for LQG are the unique non-negative definite solutions of a pair of matrix Riccati equations. Once the state deviation penalty matrix in the performance cost functional is non-singular, $R_1 > 0$, or its cost evaluation model, characterized by the pair of $(\Phi^s_k, R_1^{1/2})$, is detectable, where Φ^s_k is the discrete-time system matrix, the non-negative definite solution of the Riccati equation for calculation of control gain is assured unique. On the other hand, the uniqueness of the estimation gain (i.e. Kalman gain) is guaranteed by non-singular covariance of plant disturbance, $V_1 > 0$, or the plant disturbance model, characterized by the pair of $((\Phi^s_k)^{\mathrm{T}}, V_1^{1/2})$, is detectable. These conditions have been derived in the stochastic sense in LQCDC.

The proposed LQCDC law is potentially suitable for network-based control systems such as the future generation of aircraft which are equipped with computer networks to serve as the communications link for the vehicle management system. Further analytical research, supported by experimental verification, is needed before its acceptance as a controller design tool.

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