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# Optimal Supervisory Control of Regular Languages


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**Abstract**

This paper presents an algorithm for optimal control of regular languages, realized as deterministic finite state automata (DFSA), with (possible) penalty on event disabling. A signed real measure quantifies the behavior of controlled sublanguages based on a state transition cost matrix and a characteristic vector as reported in an earlier publication. The performance index for the proposed optimal policy is obtained by combining the measure of the supervised plant language with the cost of disabled controllable event(s). Synthesis of this optimal control policy requires at most $n$ iterations, where $n$ is the number of states of the DFSA model generated from the unsupervised regular language. The computational complexity of the optimal control synthesis is polynomial in $n$. The control algorithms are illustrated with an application example of a twin-engine surveillance aircraft.



**Index Terms**

Discrete Event Supervisory Control, Performance Measure, Formal Languages


## I. Introduction

For discrete-event supervisory control, the dynamical behavior of a physical plant is often modeled as a regular language that can be realized by a finite-state automaton [7]. The sublanguage of a controlled plant could be different under different supervisors that are constrained to satisfy different specifications. Such a partially ordered set of sublanguages requires a quantitative measure for total ordering of their respective performance. To address this issue, Wang and Ray [11] have developed a signed measure of regular languages. This work was followed by Ray and Phoha [8] and Surana and Ray [10] who have constructed a vector space of sublanguages with a metric based on the total variation measure of the language.

Several researchers have proposed optimal control of deterministic finite state automata (DFSA) based on different assumptions. Some of these researchers have attempted to quantify the controller performance using different types of cost assigned to the individual events. Passino and Antsaklis [6] proposed path costs associated with state transitions and hence optimal control of a discrete event system is equivalent to following the shortest path on the graph representing the uncontrolled system. Kumar and Garg [4] introduced the concept of payoff and control costs that are incurred only once regardless of the number of times the system visits the state associated with the cost. Consequently, the resulting cost is not a function of the dynamic behavior of the plant. Brave and Heymann [1]

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introduced the concept of optimal attractors in discrete-event control. Sengupta and Lafortune [9] used control cost in addition to the path cost in optimization of the performance index for trade-off between finding the shortest path and reducing the control cost. Although costs were assigned to the events, no distinction was made for events generated at (or leading to) different states that could be "good" or "bad". These optimal control strategies have addressed performance enhancement of discrete-event control systems without a quantitative measure of languages.

Fu et al. [3] have proposed a state-based approach to optimal control of regular languages by selectively disabling controllable events so that the resulting optimal policy can be realized as a controllable supervisor. The performance index of the optimal policy is a signed real measure of the supervised sublanguage, which is expressed in terms of a cost matrix and a characteristic vector [10], but it does not assign any additional penalty for event disabling.

This paper extends the earlier work of Fu et al. [3] on optimal control to include the cost of event disabling. The rationale is that the previously proposed optimal supervisor makes the best trade-off between reaching good states and avoiding bad states, and achieves optimal performance in terms of the language measure of the supervised plant. However, another supervisor that has a slightly inferior performance relative to the above optimal controller may only require disabling of some other controllable events, which is much less difficult to achieve. Therefore, with due consideration to event disabling, the second controller may be preferable.

From the above perspectives, the performance index for the optimal control policy proposed in this paper is obtained by combining the measure of the supervised plant language with the cost of disabled event(s). Starting with the (regular) language of an unsupervised plant automaton, the optimal control policy makes a trade-off between the measure of the supervised sublanguage and the associated event disabling cost to achieve the best performance.

The paper is organized in six sections including the present one. Section II reviews the previous work on language measure [10]. Section III presents the optimal control policy without the event disabling cost and proofs of the propositions are given in Appendix A. Section IV modifies the performance index to include the event disabling cost and formulates the algorithm of the optimal control policy with event disabling cost as an extension of Section III. Proofs of the propositions are given in Appendix B. Section V presents an application example to illustrate the concepts of optimal control without and with event disabling cost. The paper is summarized and concluded in Section VI along with recommendations for future work.

## II. Brief Review of the Language Measure

This section briefly reviews the concept of signed real measure of regular languages [11] [10]. Let the plant behavior be modeled as a deterministic finite state automaton (DFSA) as:

$$G_i \equiv (Q, \Sigma, \delta, q_i, Q_m) \tag{1}$$

where $Q$ is the finite set of states with $|Q| = n$ excluding the dump state [7] if any, and $q_i \in Q$ is the initial state; $\Sigma$ is the (finite) alphabet of events with $|\Sigma| = m$; $\Sigma*$ is the set of all finite-length strings of events including the empty string $\varepsilon$; the (possibly partial) function $\delta : Q \times \Sigma \to Q$ represents state transitions and $\delta^* : Q \times \Sigma^* \to Q$ is an extension of $\delta$; and $Q_m \subseteq Q$ is the set of marked (i.e., accepted) states.

*Definition 2.1:* The language $L(G_i)$ generated by a DFSA $G$ initialized at the state $q_i \in Q$ is defined as:

$$L(G_i) = \{s \in \Sigma^* \mid \delta^*(q_i, s) \in Q\} \tag{2}$$

*Definition 2.2:* The language $L_m(G_i)$ marked by a DFSA $G_i$ initialized at the state $q_i \in Q$ is defined as:

$$L_m(G_i) = \{s \in \Sigma^* \mid \delta^*(q_i, s) \in Q_m\} \tag{3}$$

The language $L(G_i)$ is partitioned as the non-marked and the marked languages, $L^o(G_i) \equiv L(G_i) - L_m(G_i)$ and $L_m(G_i)$, consisting of event strings that, starting from $q \in Q$, terminate at one of the non-marked states in $Q - Q_m$ and one of the marked states in $Q_m$, respectively. The set $Q_m$ is partitioned into $Q_m^+$ and $Q_m^-$, where $Q_m^+$ contains all *good* marked states that we desire to reach and $Q_m^-$ contains all *bad* marked states that we want to avoid, although it may not always be possible to avoid the bad states while attempting to reach the good states. The marked language $L_m(G)$ is further partitioned into $L_m^+(G)$ and $L_m^-(G_i)$ consisting of good and bad strings that, starting from $q_i$, terminate on $Q_m^+$ and $Q_m^-$, respectively.

A signed real measure $\mu : 2^{\Sigma^*} \to \mathrm{R} \equiv (-\infty, \infty)$ is constructed for quantitative evaluation of every event string $s \in \Sigma^*$. The language $L(G_i)$ is decomposed into null, i.e., $L^o(G_i)$, positive, i.e., $L_m^+(G_i)$, and negative, i.e., $L_m^-(G_i)$ sublanguages.

*Definition 2.3:* The language of all strings that, starting at a state $q_i \in Q$, terminates on a state $q_j \in Q$, is denoted as $L(q_i, q_j)$. That is,

$$L(q_i, q_j) \equiv \{s \in L(G_i) : \delta^*(q_i, s) = q_j\}. \tag{4}$$

*Definition 2.4:* The characteristic function that assigns a signed real weight to state-partitioned sublanguages $L(q_i, q_j), i = 1, 2, \ldots, n$ is defined as: $\chi : Q \to [-1,\ 1]$ such that

$$\chi(q_j) \in \begin{cases} [-1,\ 0) & if\, q_j \in Q_m^- \\ \{0\} & if\, q_j \notin Q_m \\ (0,\ 1] & if\, q_j \in Q_m^+ \end{cases}$$

*Definition 2.5:* The event cost is conditioned on a DFSA state at which the event is generated, and is defined as $\tilde{\pi} : \Sigma^* \times Q \to [0,\ 1]$ such that $\forall q_j \in Q, \forall \sigma_k \in \Sigma, \forall s \in \Sigma^*$,

(1) $\tilde{\pi}[\sigma_k, q_j] \equiv \tilde{\pi}_{jk} \in [0, 1)$; $\sum_k \tilde{\pi}_{jk} < 1$;
(2) $\tilde{\pi}[\sigma, q_j] = 0$ if $\delta(q_j, \sigma)$ is undefined; $\tilde{\pi}[\epsilon, q_j] = 1$;
(3) $\tilde{\pi}[\sigma_k s, q_j] = \tilde{\pi}[\sigma_k, q_j]\ \tilde{\pi}[s, \delta(q_j, \sigma_k)]$.

The event cost matrix, denoted as $\tilde{\Pi}$-matrix, is defined as:

$$\tilde{\mathbf{\Pi}} = \begin{bmatrix} \tilde{\pi}_{11} & \tilde{\pi}_{12} & \ldots & \tilde{\pi}_{1m} \\ \tilde{\pi}_{21} & \tilde{\pi}_{22} & \ldots & \tilde{\pi}_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{\pi}_{n1} & \tilde{\pi}_{n2} & \ldots & \tilde{\pi}_{nm} \end{bmatrix}$$

An application of the induction principle to part(3) in Definition 2.5 shows that

$$\tilde{\pi}[st, q_j] = \tilde{\pi}[s, q_j]\tilde{\pi}[t, \delta^*(q_j, s)]$$

The condition $\sum_k \tilde{\pi}_{jk} < 1$ provides a sufficient condition for the existence of the real signed measure as discussed in [10] along with additional comments on the physical interpretation of the event cost.

Now we define the measure of a sublanguage of the plant language $L(G_i)$ in terms of the signed characteristic function $\chi$ and the non-negative event cost $\tilde{\pi}$.

*Definition 2.6:* The signed real measure $\mu$ of a singleton string set $\{s\} \subset L(q_i, q_j) \subseteq L(G_i) \in 2^{\Sigma*}$ is defined as:

$$\mu(\{s\}) \equiv \chi(q_j)\, \tilde{\pi}(s,\ q_i) \quad \forall s \in L(q_i, q_j).$$

The signed real measure of $L(q_i, q_j)$ is defined as:

$$\mu\left(L(q_i, q_j)\right) \equiv \sum_{s \in L(q_i, q_j)} \mu\left(\{s\}\right)$$

and the signed real measure of a DFSA $G_i$, initialized at the state $q_i \in Q$, is denoted as:

$$\mu_i \equiv \mu(L(G_i)) = \sum\nolimits_j \mu\left(L(q_i, q_j)\right)$$

*Definition 2.7:* The state transition cost of the DFSA is defined as a function $\pi : Q \times Q \to [0, 1)$ *such that* $\forall q_j,\ q_k \in Q, \quad \pi(q_j,\, q_k) = \sum_{\sigma \in \Sigma:\, \delta(q_j, \sigma) = q_k} \tilde{\pi}(\sigma,\, q_j) \equiv \pi_{jk}$ *and* $\pi_{jk} = 0$ *if* $\{\sigma \in \Sigma : \delta(q_j,\, \sigma) = q_k\} = \emptyset$*. The state transition cost matrix, denoted as* $\Pi$*-matrix, is defined as:*

$$\mathbf{\Pi} = \begin{bmatrix} \pi_{11} & \pi_{12} & \dots & \pi_{1n} \\ \pi_{21} & \pi_{22} & \dots & \pi_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \pi_{n1} & \pi_{n2} & \dots & \pi_{nn} \end{bmatrix}$$

Wang and Ray [11] and Surana and Ray [10] have shown that the measure $\mu_i \equiv \mu(L(G_i))$ of the language $L(G_i)$, with the initial state $q_i$, can be expressed as: $\mu_i = \sum_j \pi_{ij}\, \mu_j \ + \chi_i$ where $\chi_i \equiv \chi(q_i)$. Equivalently, in vector notation: $\bar{\mu} = \Pi\bar{\mu} + \bar{\chi}$ where the measure vector $\bar{\mu} \equiv [\mu_1\ \mu_2\ \ \cdots\ \mu_n]^T$and the characteristic vector $\bar{\chi} \equiv [\chi_1\ \chi_2\ \ \cdots\ \ \chi_n]^T$. We delineate salient properties of the state transition cost matrix $\Pi$,which are useful for constructing the optimal control policy.

**Property 1**: *Following Definitions 4 and 6, there exists* $\theta \in (0, 1)$*such that the induced infinity norm* $\|\Pi\|_\infty \equiv \max_i \sum_j \pi_{ij} = 1 - \theta$*. The matrix operator* $[I - \Pi]$ *is invertible implying that the inverse* $[I - \Pi]^{-1}$ *is a bounded linear operator with its induced infinity norm* $||[I - \Pi]^{-1}||_\infty \leq \theta^{-1}$ [5]. *Therefore, the language measure vector can be expressed as:* $\bar{\mu} = [I - \Pi]^{-1}\bar{\chi}$*, where* $\bar{\mu} \in \mathrm{R}^n$*, and computational complexity of the measure is* $O(n^3)$ [10].

**Property 2**: *The matrix operator* $[I-\Pi]^{-1} \geq 0$ *elementwise. By Taylor series expansion,* $[I-\Pi]^{-1} = \sum_{k=0}^{\infty} [\Pi]^k$*and* $[\Pi]^k \geq 0$ *because* $\Pi \geq 0$.

**Property 3**: *The determinant* $Det\,[I - \Pi]$ *is real positive because the eigenvalues of the real matrix* $[I - \Pi]$ *appear as real or complex conjugates and they have positive real parts. Hence, the product of all eigenvalues of* $[I - \Pi]$ *is real positive.*

**Property 4**: *An affine operator* $T : \mathrm{R}^n \to \mathrm{R}^n$*can be defined as:* $T\,\bar{\nu} = \Pi\,\bar{\nu} + \bar{\chi}$ *for any arbitrary* $\nu \in \mathrm{R}^n$*. As* $\Pi$ *is a contraction,* $T$ *is also a contraction. Since* $\mathrm{R}^n$ *is a Banach space, there exists a unique fixed point of* $T$*[5] that is the measure vector* $\bar{\mu}$ *satisfying the condition* $T\,\bar{\mu} = \bar{\mu}$*. Therefore, The language measure vector* $\bar{\mu}$ *is uniquely determined as:* $\bar{\mu} = [I - \Pi]^{-1}\,\bar{\chi}$*, which can be interpreted as the unique fixed point of a contraction operator.*

## III. Optimal Control without Event Disabling Cost

This section presents the theoretical foundations of the optimal supervisory control of DFSA plants by selectively disabling controllable events so that the resulting optimal policy can be realized as a controllable supervisor [3]. The plant model is first modified to satisfy the specified operational constraints, if any. Then, starting with the (regular) language of the unsupervised plant, the optimal policy maximizes the performance of the controlled sublanguage of the supervised plant without any further constraints. The performance index of the optimal policy is a signed real measure of the supervised sublanguage, described in Section 2, which is expressed in terms of a state transition cost matrix $\Pi$ and a characteristic vector $\bar{\chi}$, , but it does not assign any additional penalty for event disabling.

Let $S \equiv \{S^0, S^1, \cdots, S^N\}$ be the finite set of all supervisory control policies that selectively disables controllable events of the unsupervised plant DFSA $G$ and can be realized as regular languages. Denoting $\Pi^k \equiv \Pi(S^k), \quad k \in \{1, 2, \cdots, N\}$, the supervisor $S^0$ is the null controller (i.e., no event is disabled) implying that $L(S^0/G) = L(G)$. Therefore the controller cost matrix $\Pi(S^0) = \Pi^0 \equiv \Pi^{plant}$ that is the $\Pi$-matrix of the unsupervised plant automaton $G$.For a supervisor $S^i$, $i \in \{1, 2, \cdots, N\}$, the control policy selectively disables certain controllable events by which the corresponding elements of the $\tilde{\Pi}$-matrix (see Definition 4) become zero. Therefore the (elementwise) inequality holds: and $L(S^k/G) \subseteq L(G) \quad \forall S^k \in S$. The language measure vector of a supervised plant $L(S^k/G)$is expressed as:

$$\bar{\mu}^k \equiv [I - \Pi^k]^{-1} \bar{\chi}$$

where the $j^{th}$ element of the vector $\bar{\mu}^k$ is denoted as $\mu_j^k$. In the sequel, $\mu^k$ is chosen to be the performance measure for the optimal control policy without event disabling cost.

*Proposition 3.1:* Let $j$ be such that $\mu_j^k = \min_{\ell \in \{1,2,\cdots,n\}} \mu_\ell^k$. If $\mu_j^k \leq 0$, then $\chi_j \leq 0$; and if $\mu_j^k < 0$, then $\chi_j < 0$.

*Corollary 3.1:* Let $\mu_j^k = \max_{\ell \in \{1,2,\cdots,n\}} \mu_\ell^k$. If $\mu_j^k \geq 0$, then $\chi_j \geq 0$ and if $\mu_j^k > 0$, *then* $\chi_j > 0$.

*Proposition 3.2:* Given $\Pi(S^k) = \Pi^k$ and $\mu^k \equiv [I - \Pi^k]^{-1}\bar{\chi}$, let $\Pi^{k+1}$ be generated from $\Pi^k$ for $k \geq 0$ by disabling or re-enabling the appropriate controllable events as follows: $\forall i, j \in \{1, 2, \cdots, n\}, \quad ij^{th}$ *element of* $\Pi^{k+1}$ is modified as:

$$\pi_{ij}^{k+1} \begin{cases} \geq \pi_{ij}^k & if\ \mu_j^k > 0 \\ = \pi_{ij}^k & if\ \mu_j^k = 0 \\ \leq \pi_{ij}^k & if\ \mu_j^k < 0 \end{cases} \tag{5}$$

and $\Pi^k \leq \Pi^0 \ \forall k$. Then, $\bar{\mu}^{k+1} \geq \bar{\mu}^k$ elementwise and equality holds if and only if $\Pi^{k+1} = \Pi^k$.

*Corollary 3.2:* For a given state $q_j$, let $\mu_j^k < 0$ and $\Pi^{k+1}$ be generated from $\Pi^k$ by disabling controllable events that lead to the state $q_j$. Then, $\mu_j^{k+1} < 0$.

In Proposition 3.2, some elements of the $j^{th}$ column of $\Pi^k$ are decreased (or increased) by disabling (or re-enabling) controllable events that lead to the states $q_j$ for which $\mu_j^k < 0$ (or $\mu_j^k \geq 0$). Next we show that an optimal supervisor can be achieved to yield best performance in terms of the language measure.

*Proposition 3.3:* Iterations of event disabling and re-enabling lead to a cost matrix $\Pi^*$ that is optimal in the sense of maximizing the performance vector $\bar{\mu}^* \equiv [I - \Pi^*]^{-1}\bar{\chi}$ elementwise.

*Proposition 3.4:* The control policy induced by the optimal $\Pi^*$-matrix in Proposition 3.3 is unique in the sense that the controlled language is most permissive (i.e., least restrictive) among all controller(s) having the best performance.

Propositions 3.3 and 3.4 suffice to conclude that the $\Pi^*$-matrix yields the most permissive controller with the best performance $\bar{\mu}^*$. The optimal control policy (without event disabling cost) can be realized as:

- All controllable events leading to the states $q_j$, for which $\mu_j^* < 0$, are disabled;
- All controllable events leading to the states $q_j$, for which $\mu_j^* \geq 0$, are enabled.

*A. Construction of the Optimal Control Policy without Event Disabling Cost*

We propose a procedure for construction of the optimal control policy that maximizes the performance of the controlled language of the DFSA (without event disabling cost), starting from any initial state $q \in Q$. Let $G$ be a DFSA plant model without any constraint (i.e., operational specifications) and have the state transition cost matrix of the open loop plant as: $\Pi^{plant} \in \Re^{n \times n}$ and the characteristic vector as: $\bar{\chi} \in \Re^n$. Then, the performance vector at $k = 0$ is given as: $\bar{\mu}^0 = [\mu_1^0 \; \mu_2^0 \; \cdots \; \mu_n^0]^T = (I - \Pi^0)^{-1} \bar{\chi}$, where the $j^{th}$ element $\mu_j^0$ of the vector $\mu^0$ is the performance of the language, with state $q_j$ as the initial state. Then, $\mu_j^0 < 0$ implies that, if the state $q_j$ is reached, then the plant will yield bad performance thereafter. Intuitively, the control system should attempt to prevent the automaton from reaching $q_j$ by disabling all controllable events that lead to this state. Therefore, the optimal control algorithm starts with disabling all controllable events that lead to every state $q_j$ for which $\mu_j^0 < 0$. This is equivalent to reducing all elements of the corresponding columns of the $\Pi^0$-matrix by disabling those controllable events. In the next iteration, i.e., $k = 1$, the updated cost matrix $\Pi^1$ is obtained as: $\Pi^1 = \Pi^0 - \Delta^0$ where $\Delta^0 \geq 0$ (the inequality being implied elementwise) is composed of event costs corresponding to all controllable events that have been disabled. Using Proposition 2, $\bar{\mu}^0 \leq \bar{\mu}^1 \equiv [I - \Pi^1]^{-1} \bar{\chi}$. Although all controllable events leading to every state corresponding to a negative element of $\mu^1$ are disabled, some of the controllable events that were disabled at $k = 0$ may now lead to states corresponding to positive elements of $\mu^1$. Performance could be further enhanced by re-enabling these controllable events. For $k \geq 1$, $\Pi^{k+1} = \Pi^k + \Delta^k$ where $\Delta^k \geq 0$ is composed of the state transition costs of all re-enabled controllable events at $k$.

If $\bar{\mu}^0 \geq 0$, i.e., there is no state $q_j$ such that $\mu_j^0 < 0$, then the plant performance cannot be improved by event disabling and the null controller $S^0$ (i.e., no disabled event) is the optimal controller for the given plant. Therefore, we consider the cases where $\mu_j^0 < 0$ for some state $q_j$.

Starting with $k = 0$ and $\Pi^0 \equiv \Pi^{plant}$, the control policy is constructed by the following two-step procedure:

***Step 1***: For every state $q_j$ for which $\mu_j^0 < 0$, disable controllable events leading to $q_j$. Now, $\Pi^1 = \Pi^0 - \Delta^0$, where $\Delta^0 \geq 0$ is composed of event costs corresponding to all controllable events, leading to $q_j$ for which $\mu_j^0 < 0$, which have been disabled at $k = 0$.

***Step 2***: For $k \geq 1$, if $\mu_j^k \geq 0$, re-enable all controllable events leading to $q_j$, which were disabled in Step 1. The cost matrix is updated as: $\Pi^{k+1} = \Pi^k + \Delta^k$ for $k \geq 1$, where $\Delta^k \geq 0$ is composed of event costs corresponding to all currently re-enabled controllable events. The iteration is terminated if no controllable event leading to $q_j$ remains disabled for which $\mu_j^k > 0$. At this stage, the optimal performance $\bar{\mu}^* \equiv [I - \Pi^*]^{-1} \bar{\chi}$.

*Proposition 3.5:* The number of iterations needed to arrive at the optimal control law without event disabling cost does not exceed the number, $n$, of states of the DFSA.

Since each iteration in the synthesis of the optimal control requires a single Gaussian elimination of $n$ unknowns from $n$ linear algebraic equations, computational complexity of the control algorithm is polynomial in $n$.

## IV. OPTIMAL CONTROL WITH EVENT DISABLING COST

This section presents the optimal control policy with event disabling cost by including the cost of all (controllable) events, disabled by the supervisor, in the performance cost. As the cost of disabled event(s) approaches zero, the optimal control policy with event disabling cost converges to the optimal control policy without event disabling cost, described in Section .

*Definition 4.1:* Let the cost of disabling a (controllable) event $\sigma_j$ that causes transition from $q_i$ be denoted as $c_{ij}$ where $c_{ij} \in [0, 1]$. The $(n \times m)$ disabling cost matrix is denoted as $C = [c_{ij}]$.

Since the (controllable) supervisor never disables any uncontrollable event, the entries $c_{ij}$for uncontrollable events have no importance. For implementation, they can be set to an arbitrarily large positive $M < \infty$.

*Definition 4.2:* The action of disabling a (controllable) event $\sigma_j$ at state $q_i$ by a supervisor $S$ is defined as:

$$d_{ij}^S = \begin{cases} 1 \;\; if \; \sigma_j \text{ is disabled at state } q_i \\ 0 \quad \text{otherwise} \end{cases}$$

*The* $(n \times m)$ action matrix of disabling controllable events by a supervisor $S$ is denoted as: $D^S = [d_{ij}^S]$.

*Definition 4.3:* The event disabling cost characteristic of a supervisor $S$ that selectively disables controllable events $\sigma_j$ at state $q_i$ is defined as:

$$\gamma_i^S = \sum_{j:\, d_{ij}^S = 1} c_{ij}\, \tilde{\pi}_{ij}$$

The disabling cost characteristic is proportional to event cost of the controllable event disabled by the supervisor $S$.

The $(n \times 1)$ disabling cost characteristic vector of a supervisor $S$ is denoted as: $\bar{\gamma}^s \equiv [\gamma_1^S \; \gamma_2^S \; \cdots \; \gamma_n^S]^T$.

*Definition 4.4:* The modified characteristic of a state $q_i \in Q$ is defined as:

$$\chi_i^s \equiv \chi_i - \gamma_i^s.$$

The $(n \times 1)$ modified characteristic vector under a supervisor $S$ is defined as:

$$\bar{\chi}^s \equiv \bar{\chi} - \bar{\gamma}^s$$

where $\bar{\chi}^s \equiv [\chi_i^s \; \chi_i^s \; \cdots \; \chi_n^s]^T$.

*Definition 4.5:* The disabling cost measure vector under a supervisor $S$ is defined as:

$$\bar{\theta}^s \equiv [I - \Pi^S]^{-1} \bar{\gamma}^s.$$

with $\theta_i^s$ being the $i^{th}$ element of $\bar{\theta}^s$, which is the disabling cost incurred by the state.

*Definition 4.6:* The performance measure vector of a supervisor $S$is defined as:

$$\bar{\eta}^s \equiv [I - \Pi^s]^{-1} \bar{\chi}^s$$

with $\eta_i^s$ being the $i^{th}$ element of $\bar{\eta}^s$.

The performance index vector $\bar{\eta}^s$of a supervisor $S$ can be interpreted as the difference between the measure vector $\bar{\mu}^s$of the supervised language $L(S/G)$ of the DFSA $G$ and the respective disabling cost measure vector $\bar{\theta}^s$. That is,

$$\bar{\eta}^s = \bar{\mu}^s - \bar{\theta}^s.$$

Following the approach taken for optimal control without event disabling cost in Section 3, let $S \equiv \{S^0, S^1, \cdots, S^N\}$ be the finite set of supervisory control policies that can be realized as regular languages. For a supervisor $S^k \in S$, the control policy selectively disables certain controllable events. Consequently, the corresponding elements of the $\tilde{\Pi}$-matrix become zero and those of the event disabling characteristic vector $\bar{\gamma}^s$ are entered in the modified characteristic vector $\bar{\chi}^s$ as seen in Definition 10; therefore, $L(S^k/G) \subseteq L(G) \forall S^k \in S$. Denoting $\Pi^k \equiv \Pi(S^k), \quad k \in \{1, 2, \cdots, N\}$, the performance measure vector (with event disabling cost) of the supervised plant $L(S^k/G)$is expressed as:

$$\bar{\eta}^k \equiv [I - \Pi^k]^{-1} (\bar{\chi} - \bar{\gamma}^k)$$

where $\bar{\eta}^k \equiv \bar{\eta}^{s^k}$ and $\bar{\gamma}^k \equiv \bar{\gamma}^{s^k}$; and the $j^{th}$ element of the vector $\bar{\eta}^k$ is denoted as $\eta_j^k$. The null supervisor $S^0$ (i.e., no disabled event) has zero disabling cost, i.e., $\bar{\gamma}^0 = 0$ and consequently $\bar{\eta}^0 = \bar{\mu}^0$. We extend the optimal policy construction to include the event disabling cost.

### *A. Construction of the Optimal Control Policy with Event Disabling Cost*

This subsection formulates an optimal control policy with event disabling cost, which maximizes all elements of the performance vector $\bar{\eta}^s$ of the supervised language of a DFSA $G$ with event cost matrix $\tilde{\Pi} \in \Re^{n\times m}$; state transition cost matrix $\Pi \in \Re^{n\times n}$; characteristic vector $\bar{\chi} \in \Re^n$; and the disabling cost matrix $C \in \Re^{n\times m}$. For the unsupervised plant, we set $\Pi^0 \equiv \Pi^{plant}$; $\bar{\chi}^0 = \bar{\chi}$; $\bar{\gamma}^0 = 0$; $D^0 = 0$ (no event disabled so far). For optimal control without event disabling cost in Section 3.1, we disable all controllable events leading to states $q_\ell$ for which $\mu_\ell^0 < 0$ and subsequently, for $k \geq 1$, re-enable all previously disabled controllable events leading to $q_j$ if $\mu_j^k \geq 0$. In contrast, for optimal control with event disabling cost, we disable all controllable events $\sigma_j$ leading to states $q_\ell$ for which $\eta_\ell^0 < -c_{ij}$ with $\delta(q_i, \sigma_j) = q_\ell$, and subsequently, for $k \geq 1$, re-enable these disabled events if $\eta_\ell^k \geq -c_{ij}$. The rationale is that disabling of states with small negative performance may not be advantageous because of incurring additional event disabling cost.

The control policy with event disabling cost is formulated according to the following two-step procedure:

***Step 1***: Starting at $k = 0$, disable all controllable events $\sigma_j$, leading to each state $q_\ell$ if the inequality: $\eta_\ell^0 < -c_{ij}$ with $\delta(q_i, \sigma_j) = q_\ell$ is satisfied. The algorithm for dealing with this inequality is delineated below:

• If the inequality is not satisfied for any single case, stop the iterative procedure. No event disabling can improve the plant performance beyond that of the open loop plant, i.e., the null supervisor $S^0$ achieves optimal control.

• If the inequality is satisfied for at least one case, disable the qualified event(s) and update the state transition cost matrix to $\Pi^1 \leq \Pi^0$ (elementwise); the disabling matrix to $D^1$ for generating the cost characteristic function $\bar{\gamma}^1$; and the modified characteristic vector $\bar{\chi}^1 \equiv \bar{\chi} - \bar{\gamma}^1$. Go to Step 2.

***Step 2***: The performance measure vector for $k \geq 1$ is

$$\bar{\eta}^k \equiv [I - \Pi^k]^{-1}\bar{\chi}^k = [I - \Pi^k]^{-1}(\bar{\chi} - \bar{\gamma}^k)$$

, re-enable all previously (at $k = 0$) disabled controllable events $\sigma_j$, leading to states $q_\ell$ if the inequality $\eta_\ell^k \geq -c_{ij}$with $\delta(q_i, \sigma_j) = q_\ell$ is satisfied. The algorithm for dealing with this inequality is as follows:

- If the inequality is not satisfied for any single case, an optimal control is achieved and the iterative procedure is complete. No further event re-enabling can improve the controlled plant performance beyond that of the current supervisor that is the optimal controller.
- If the inequality is satisfied for at least one case, re-enable all qualified events and update the state transition cost matrix to $\Pi^{k+1} \geq \Pi^k$ (elementwise); the disabling matrix to $D^k$; the cost characteristic function to $\bar{\gamma}^{k+1}$; and the modified characteristic vector $\bar{\chi}^{k+1} \equiv \bar{\chi} - \bar{\gamma}^{k+1}$. Update $k \leftarrow (k+1)$ and repeat Step 2 until the inequality $\eta_\ell^k \geq -c_{ij}$ with $\delta(q_i, \sigma_j) = q_\ell$ is not satisfied for all $j$ and $\ell$. Then, the current supervisor is optimal in terms of the performance measure in Definition 12.

The above procedure for optimal control with event disabling cost is an extension of that without event disabling cost described in Section 3.1. For zero event disabling cost, the two procedures become identical. Following the rationale of Proposition 5, the computational complexity of the control synthesis with disabling cost is also polynomial in $n$.

We present the underlying theory of unconstrained optimal control with event disabling cost as two new propositions, which simultaneously maximize all elements of the performance vector $\bar{\eta}$.

*Proposition 4.1:* For all supervisors $S^k$ in the iterative procedure, $\bar{\eta}^{k+1} \geq \bar{\eta}^k$ elementwise.

*Proposition 4.2:* The supervisor $S$ generated upon completion of the algorithm in Section IV is optimal in terms of the performance in Definition 4.6.

## V. Example of Discrete Event Optimal Supervisory Control

This section presents an example of the above discrete-event optimal control policies for the design of discrete-event optimal supervisors for a twin-engine unmanned aircraft that is used for surveillance and data collection. Engine health and operating conditions, which are monitored in real time based on avionic sensor information, are classified into three mutually exclusive and exhaustive categories: *good*; *unhealthy* (but operable); and *inoperable*. Upon occurrence of any observed abnormality, the supervisor decides to continue or abort the mission.

The control objective is to enhance engine safety operation. Engine health and operating conditions, which are monitored in real time based on avionic sensor information, are classified into three mutually exclusive and exhaustive categories: (i) *good*; (ii) *unhealthy (but operable)*; and (iii) *inoperable*. Upon occurrence of any observed abnormality, the supervisor decides to continue or abort the mission.

TABLE I
PLANT AUTOMATON STATES

| State | Description |
|---|---|
| 1 | Safe in base |
| 2 | Mission executing - two good engines |
| 3 | One engine unhealthy during mission execution |
| 4 | Mission executing - one good and one unhealthy engine |
| 5 | Both engines unhealthy during mission execution |
| 6 | One engine good and one engine inoperable |
| 7 | Mission execution with two unhealthy engines |
| 8 | Mission execution with only one good engine |
| 9 | One engine unhealthy and one engine inoperable |
| 10 | Mission execution with only one unhealthy engine |
| 11 | Mission aborted /not completed (Bad Marked State) |
| 12 | Mission successful (Good Marked State) |
| 13 | Aircraft destroyed (Bad Marked State) |

TABLE II
PLANT EVENT ALPHABET

| Event | Event Description | Controllable Events |
|---|---|---|
| $s$ | start and take-off | $\surd$ |
| $b$ | a good engine becoming unhealthy | |
| $t$ | an unhealthy engine becoming inoperable | |
| $v$ | a good engine becoming inoperable | |
| $k$ | keep engine(s) running | $\surd$ |
| $a$ | mission abortion | $\surd$ |
| $f$ | mission completion | |
| $d$ | destroyed aircraft | |
| $l$ | landing | $\surd$ |

The deterministic finite state automaton model of the (unsupervised) plant (i.e., engine operation) has 13 states, of which three are marked (i.e., accepted) states, and nine events, of which four are controllable. The dump state is not included as it is not of interest in the supervisory control synthesis

[7] [3]. All events are assumed to be observable. The states and events of the plant model are listed in Table I and Table II, respectively. As indicated in Table I, the marked states are: 11, 12 and 13, of which the states 11 and 13 are bad marked states, and the state 12 is a good marked state.

TABLE III

STATE TRANSITION $\delta$ EVENT COST $\tilde{\Pi}$ AND DISABLING COST $C$ MATRICES

| | $s$ | $b$ | $t$ | $v$ | $k$ | $a$ | $f$ | $d$ | $l$ |
|---|---|---|---|---|---|---|---|---|---|
| 1 | (2)<br>0.500<br>0.000 | | | | | (1)<br>0.020<br>0.005 | | | |
| 2 | | (3)<br>0.050<br>N/A | | (6)<br>0.010<br>N/A | | | (12)<br>0.800<br>N/A | (3)<br>0.100<br>N/A | |
| 3 | | | | | (4)<br>0.450<br>0.050 | (11)<br>0.450<br>0.005 | | | |
| 4 | | (5)<br>0.120<br>N/A | (6)<br>0.160<br>N/A | (9)<br>0.100<br>N/A | | | (12)<br>0.500<br>N/A | (13)<br>0.120<br>N/A | |
| 5 | | | | | (7)<br>0.450<br>0.080 | (11)<br>0.450<br>0.002 | | | |
| 6 | | | | | (8)<br>0.450<br>0.010 | (11)<br>0.450<br>0.004 | | | |
| 7 | | | (9)<br>0.250<br>N/A | | | | (12)<br>0.500<br>N/A | (13)<br>0.200<br>N/A | |
| 8 | | (9)<br>0.200<br>N/A | | (13)<br>0.010<br>N/A | | | (12)<br>0.300<br>N/A | (13)<br>0.400<br>N/A | |
| 9 | | | | | (10)<br>0.450<br>0.35 | (11)<br>0.450<br>0.002 | | | |
| 10 | | | (13)<br>0.350<br>N/A | | | | (12)<br>0.200<br>N/A | (13)<br>0.400<br>N/A | |
| 11 | | | | | | | | | (1)<br>0.95<br>0.000 |
| 12 | | | | | | | | | (1)<br>0.95<br>0.000 |
| 13 | | | | | | | | | |

Characteristic Vector $\bar{\chi} = [0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ \ -0.05\ \ 0.25\ \ -1.0]^T$
(See Definition 2.4)

The state transition function $\delta$ (see the beginning of Section II), the entries $\tilde{\pi}_{ij}$ (see Definition 2.4) of the event cost matrix $\tilde{\Pi}$, and the entries $c_{ij}$ (see Definition 4.1) of the event disabling cost matrix $C$ are entered simultaneously in relevant cells of Table III. The dump state and any transitions to the dumped state are not shown in Table III. The empty cells in Table III imply that the state transition function $\delta$ is undefined for the respective state and event. In each non-empty cell in Table III, the positive integer in the first entry signifies the destination state of the transition; the non-negative fraction in the second entry is the state-based event cost $\tilde{\pi}_{ij}$; and the non-negative fraction in the third entry is the state-based event disabling cost $c_{ij}$ of the four controllable events (i.e., events s, k, a and $\ell$); event disabling cost is not applicable to the remaining five uncontrollable events (i.e., events b, t, v, f and d) and the corresponding entries are marked as "N/A". (Note that the event cost $\tilde{\pi}_{ij}$ and event disabling cost $c_{ij}$ of a given event could be different at different states.)

The values of $\tilde{\pi}_{ij}$ were selected by extensive simulation experiments on gas turbine engine models and were also based on experience of gas turbine engine operation and maintenance. The state-based event cost $\tilde{\pi}_{ij}$ such that each row sum of the event cost matrix $\tilde{\Pi}$ is strictly less than one as given in Definition 2.5 and explained in detail by in a previous publication [10]. The event disabling cost $c_{ij}$ for controllable events indicates the difficulty of disabling from the respective states and the values were chosen based on operational experience. The elements of the characteristic vector (see Definition 2.4) are chosen as non-negative weights based on the perception of each marked state's role on the gas turbine system performance. In this simulation example, the characteristic value of the good marked state 12 is taken to be 0.25 and those of the bad marked states 11 and 13 are taken to be –0.05 and

–1.0, respectively, to quantify their respective importance; each of the remaining non-marked states is assigned zero characteristic value as seen at the bottom of Table III. The information provided in Table III is sufficient to generate the state transition cost matrix $\Pi$ (see Definition 2.7).

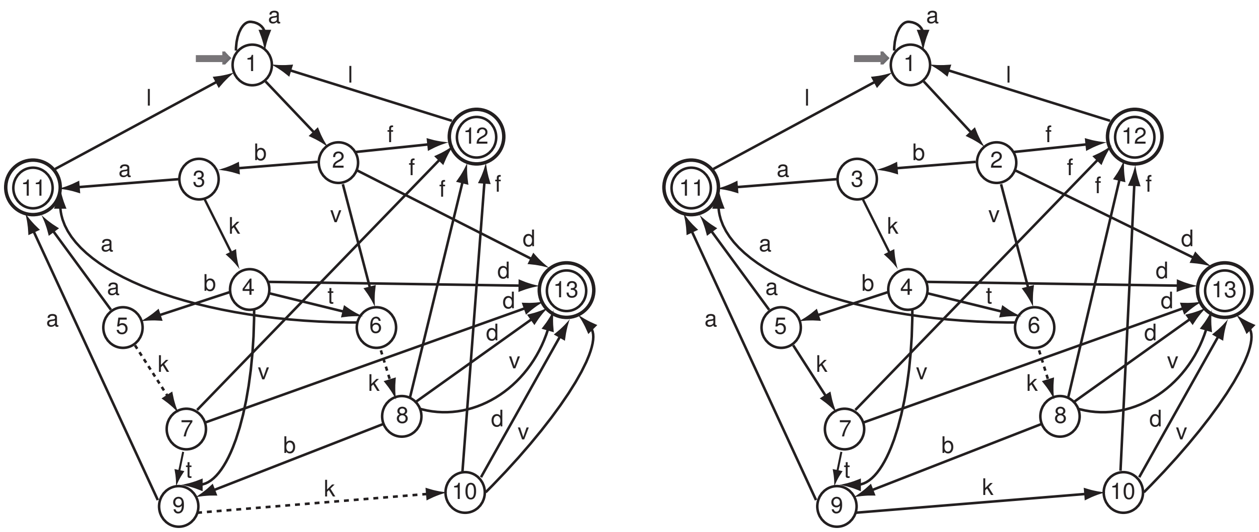
TABLE IV
SYNTHESIS WITHOUT EVENT DISABLING COST

| Iteration 0 | Iteration 1 | Iteration 2 |
|---|---|---|
| 0.0823 | 0.0840 | 0.0850 |
| 0.1613 | 0.1646 | 0.1665 |
| 0.0062 | 0.0134 | 0.0366 |
| -0.0145 | 0.0500 | 0.0506 |
| -0.0367 | 0.0134 | 0.0138 |
| -0.1541 | 0.0134 | 0.0138 |
| -0.1097 | -0.0317 | -0.0312 |
| -0.3706 | -0.3084 | -0.3080 |
| -0.2953 | 0.0134 | 0.0138 |
| -0.6844 | -0.6840 | -0.6839 |
| 0.0282 | 0.0298 | 0.0307 |
| 0.3282 | 0.3298 | 0.3307 |
| -1.0000 | -1.0000 | -1.0000 |

TABLE V
SYNTHESIS WITH EVENT DISABLING COST

| Iteration 0 | Iteration 1 | Iteration 2 |
|---|---|---|
| 0.0823 | 0.0839 | 0.0841 |
| 0.1613 | 0.1645 | 0.1649 |
| 0.0062 | 0.0134 | 0.0188 |
| -0.0145 | 0.0117 | 0.0118 |
| -0.0367 | -0.0356 | -0.0354 |
| -0.1541 | 0.0034 | 0.0035 |
| -0.1097 | -0.1088 | -0.1086 |
| -0.3706 | -0.3700 | -0.3699 |
| -0.2953 | -0.2944 | -0.2943 |
| -0.6844 | -0.6841 | -0.6840 |
| 0.0282 | 0.0297 | 0.0299 |
| 0.3282 | 0.3297 | 0.3299 |
| -1.0000 | -1.0000 | -1.0000 |

Based on the data given in Tables I, II and III, two optimal control policies - Case (a) without event disabling cost and the other Case (b) with event disabling cost have been synthesized following the respective two-step procedures in Sections III and IV. The results of optimal supervisor syntheses without and with event disabling cost are presented in Tables IV and V supported by respective finite state machine diagrams in Figures 1(a) and 1(b). For Case(a), the event disabling cost matrix $C$ (i.e., the relevant elements in Table III) are set to zero for synthesis of the optimal control without event disabling cost. In contrast, for Case (b), all elements the event disabling cost matrix $C$ in Table III are used for synthesis of the optimal control with event disabling cost. At successive iterations, Table IV lists the performance vectors in Case (a): $\bar{\mu}^0$ for the unsupervised (i.e., open loop) plant, $\bar{\mu}^1$ in iteration 1, and $\bar{\mu}^2$ in iteration 2 when the synthesis is completed because of no sign change between elements of $\bar{\mu}^1$ and $\bar{\mu}^2$. Table IV shows that $\bar{\mu}^2 \geq \bar{\mu}^1 \geq \bar{\mu}^0$ elementwise. This is due to disabling the controllable event $k$ leading to states 7, 8 and 10 as indicated by the dashed arcs in the state transition




(a) Supervision without Event Disabling Cost (b) Supervision with Event Disabling Cost

Fig. 1. Finite State Machine Diagrams of Optimally Supervised Systems

diagram of Figure 1(a). Consequently, the states 7, 8, and 10 become isolated as there are no other events leading to these states. Starting with the initial state 1, indicated by an external arrow in Figure 1(a), the optimal performance is 0.0850 that is the first element $\mu_1^2$ of the performance vector $\bar{\mu}^2$ as seen in the top right hand corner in Table IV.

The results are different for Case (b) because the event disabling cost is taken into account in optimal supervisor synthesis as seen in Table V and Figure 1(b); in this case, only the state 8 is isolated due to disabling of the controllable event $k$ at the state 6. At successive iterations, Table V lists the performance vectors for this Case (b) where $\bar{\eta}^0 = \bar{\mu}^0$ for the unsupervised (i.e., open loop) plant; $\bar{\eta}^1$ in iteration 1, and $\bar{\eta}^2$ in iteration 2 when the synthesis is completed because of no sign change between elements of $\bar{\eta}^1$ and $\bar{\eta}^2$. (Note that, in general, the number of iterations needed for supervisor synthesis without and with event disabling cost may not be the same.) Table V shows that $\bar{\eta}^2 \geq \bar{\eta}^1 \geq \bar{\eta}^0$ elementwise. This is due to disabling of the controllable event k leading to the state 8 as indicated by the dashed arcs in the state transition diagram of Figure 1(b). Consequently, the state 8 (shown in a dotted circle in Figure 1 (b)) becomes isolated as there are no other events leading to this state. Starting with the initial state 1, indicated by an arrow in Figure 1(b), the optimal performance is 0.0841 that is the first element $\bar{\eta}_1^2$ of the performance vector $\bar{\eta}^2$ as seen in the top right hand corner in Table V. Clearly, the performance of the supervisor in Case (b) is suboptimal with respect to Case (a). That is, the performance in Case (b) cannot excel that in Case (a)) where the event disabling cost is not taken into account.

## VI. Summary and Conclusions

This paper presents the theory, formulation, and validation of optimal supervisory control policies for dynamical systems, modeled as deterministic finite state automata (DFSA), which may have already been subjected to constraints such as control specifications. The synthesis procedure for optimal control without and with event disabling cost is quantitative and relies on a signed real measure of regular languages, which is based on a specified state transition cost matrix and a characteristic vector [10].

The state-based optimal control policy without event disabling cost maximizes the language measure vector $\bar{\mu}$ by attempting to selectively disable controllable events that may lead to bad marked states and simultaneously ensuring that the remaining controllable events are kept enabled. The goal is to maximize the measure of the controlled plant language without any further constraints. The control policy induced by the updated state transition cost matrix yields maximal performance and is unique in the sense that the controlled language is most permissive (i.e., least restrictive) among all controller(s) having the optimal performance.

The performance measure vector $\bar{\eta}$, for optimal control with disabling cost, is obtained as the language measure vector of the supervised plant minus the disabling cost characteristic vector. The optimal control policy maximizes the performance vector elementwise by attempting to avoid termination on bad marked states by selectively disabling controllable events with reasonable disabling costs, and simultaneously ensuring that the remaining controllable events are kept enabled. As the cost of event disabling approaches zero, the optimal control policy with event disabling cost converges to that without event disabling cost.

Derivation of the optimal supervisory control policies requires at most $n$ iterations, where $n$ is the number of states of the DFSA model and each iteration is required to solve a set of $n$ simultaneous linear algebraic equations having complexity of $O(n^3)$[10]. As such computational complexity of the control synthesis procedure is polynomial in the number of DFSA model states. The procedure for synthesis of the optimal control algorithm has been validated on the DFSA model of a twin-engine surveillance aircraft.

Future areas of research in optimal control include robustness of the control policy relative to unstructured and structured uncertainties in the plant model including variations in the language measure parameters [2].

## APPENDIX A.
## PROOFS OF PROPOSITIONS: OPTIMAL CONTROL WITHOUT EVENT DISABLING COST

This appendix presents the proofs of five propositions and two crollaries, presented in Section III.

**Proof of Proposition 3.1**: The DFSA satisfies the identity $\mu_j^k = \sum_{\ell \in \{1,2,\cdots,n\}} \pi_{j\ell}^k \, \mu_\ell^k + \chi_j$ that leads to the inequality $\mu_j^k \geq (\sum_\ell \pi_{j\ell}^k) \, \mu_j^k + \chi_j \Rightarrow \quad (1 - \sum_\ell \pi_{j\ell}^k) \, \mu_j^k \geq \chi_j$. The proof follows from $(1 - \sum_\ell \pi_{j\ell}^k) > 0$ (see Definitions 2.5 and 2.7).

**Proof of Corollary to Proposition 3.1**: The proof is similar to that of Proposition 3.1.

**Proof of Proposition 3.2**: It follows from the the properties of the measure vector $\bar{\mu}$ that:

$$\begin{aligned}
\bar{\mu}^{k+1} - \bar{\mu}^k &= \left( \left[ I - \Pi^{k+1} \right]^{-1} - \left[ I - \Pi^k \right]^{-1} \right) \bar{\chi} \\
&= \left[ I - \Pi^{k+1} \right]^{-1} \left( [I - \Pi^k] - [I - \Pi^{k+1}] \right) \left[ I - \Pi^k \right]^{-1} \bar{\chi} \\
&= \left[ I - \Pi^{k+1} \right]^{-1} \left( \Pi^{k+1} - \Pi^k \right) \bar{\mu}^k
\end{aligned}$$

Defining the matrix $\Delta^k \equiv \Pi^{k+1} - \Pi^k$, let the $j^{th}$ column of $\Delta^k$ be denoted as $\Delta_j^k$. Then, $\Delta_j^k \leq 0$ if $\mu_j^k < 0$ and $\Delta_j^k \geq 0$ if $\mu_j^k \geq 0$, and the remaining columns of $\Delta^k$ are zero vectors. This implies that: $\Delta^k \bar{\mu}^k = \sum_j \Delta_j^k \, \mu_j^k \geq 0$. Since $\Pi^k \leq \Pi^0 \; \forall k$, $[I - \Pi^{k+1}]^{-1} \geq 0$ elementwise. Then, it follows that $[I - \Pi^{k+1}]^{-1} \Delta^k \bar{\mu}^k \geq 0 \Rightarrow \bar{\mu}^{k+1} \geq \bar{\mu}^k$. For $\mu_j^k \neq 0$ and $\Delta^k$ as defined above, $\Delta^k \bar{\mu}^k = 0$ if and only if $\Delta^k = 0$. Then, $\Pi^{k+1} = \Pi^k$ and $\bar{\mu}^{k+1} = \bar{\mu}^k$.

**Proof of Corollary to Proposition 3.2**: Since only $j^{th}$ column of $[I - \Pi^{k+1}]$ is different from that of $[I - \Pi^k]$ and the remaining columns are the same, the $j^{th}$ row of the cofactor matrix of $[I - \Pi^{k+1}]$ is the same as that of the cofactor matrix of $[I - \Pi^k]$. Therefore,

$$Det\,[I - \Pi^{k+1}] \mu_j^{k+1} = Det\,[I - \Pi^k] \mu_j^k$$

Since both determinants are real positive by Property 5 of the $\Pi$-matrix, $\mu_j^k$ and $\mu_j^{k+1}$ have the same sign.

**Proof of Proposition 3.3**: Let us consider an arbitrary cost matrix $\tilde{\Pi} \leq \Pi^0$ and $\tilde{\bar{\mu}} \equiv [I - \tilde{\Pi}]^{-1}\bar{\chi}$. It will be shown that $\tilde{\bar{\mu}} \leq \bar{\mu}^*$. Let us rearrange the elements of the $\bar{\mu}^*$-vector such that $\bar{\mu}^* = [\underbrace{\mu_1^* \cdots \mu_\ell^*}_{\geq 0} \mid \underbrace{\mu_{\ell+1}^* \cdots \mu_n^*}_{<0}]^T$ and the cost matrices $\tilde{\Pi}$ and $\Pi^*$ are also rearranged in the order in which the $\bar{\mu}^*$-vector is arranged.

According to Proposition 3.2, no controllable event leading to states $q_k$, $k = 1,\ 2,\ \cdots,\ \ell$, is disabled and all controllable events leading to states $q_k, \quad k = \ell+1,\ \ell+2,\ \cdots,\ n$, are disabled. Therefore, the elements in the first $\ell$ columns of $\Pi^*$ are the same as those of the $\Pi^0$ and only the elements in the last $(n-\ell)$ columns are decreased to the maximum permissible extent by disabling all controllable events. In contrast, the columns of $\tilde{\Pi}$ are reduced by an arbitrary choice. Therefore, defining $\Delta\Pi^* \equiv [\tilde{\Pi} - \Pi^*]$, the first $\ell$ columns of $\Delta\Pi \ \leq 0$ and the last $(n-\ell)$ columns of $\Delta\Pi \ \geq 0$.

Since $\bar{\mu}^* = [\underbrace{\mu_1^* \cdots \mu_\ell^*}_{\geq 0} \mid \underbrace{\mu_{\ell+1}^* \cdots \mu_n^*}_{<0}]^T$ and $[I - \tilde{\Pi}]^{-1} \geq 0$ elementwise, and $\tilde{\bar{\mu}} - \bar{\mu}^* = [I - \tilde{\Pi}]^{-1}\ [\tilde{\Pi} - \Pi^*]\ \mu^*$, it follows that

$$\tilde{\bar{\mu}} - \bar{\mu}^* = \underbrace{[I - \tilde{\Pi}]^{-1}}_{\geq 0} \left( \underbrace{\sum_{j=1}^{\ell} Col_j \cdot \mu_j^*}_{\leq 0} + \underbrace{\sum_{j=\ell+1}^{n} Col_j \cdot \mu_j^*}_{\leq 0} \right) \leq 0$$

Therefore, $\tilde{\bar{\mu}} \leq \bar{\mu}^*$ for any arbitrary choice of $0 \leq \tilde{\Pi} \leq \Pi^0$.

**Proof of Proposition 3.4**: Disabling controllable event(s) leading to a state $q_j$ with performance measure $\mu_j^* = 0$ does not alter the performance vector $\bar{\mu}^*$. The optimal control does not disable any controllable event leading to the states with zero performance. This implies that, among all controllers with the identical performance $\bar{\mu}^*$, the control policy induced by the $\Pi^*$-matrix is most permissive.

**Proof of Proposition 3.5**: Following Proposition 3.2, the sequence of performance vectors $\{\Pi^k\}$ in successive iterations of the two-step procedure is monotonically increasing. The first iteration at $k = 0$ disables controllable events following Step 1 of the two-step procedure in Section III-A. During each subsequent iteration in Step 2, the controllable events leading to at least one state are re-enabled. When Step 2 is terminated, there remains at least one negative element, $\mu_j^k < 0$ by 3.2. Therefore, as the number of iterations in Step 2 is at most $n-1$, the total number of iterations to complete the two-step procedure does not exceed $n$.

## APPENDIX B.
## PROOFS OF PROPOSITIONS: OPTIMAL CONTROL WITH EVENT DISABLING COST

This appendix presents the proofs of two propositions, presented in Section IV.

**Proof of Proposition 4.1**: Given $\bar{\chi}^k \equiv \bar{\chi} - \bar{\gamma}^k$ and $\bar{\eta}^k \equiv [I - \Pi^k]^{-1}\bar{\chi}^k$, let us denote the change in event disabling characteristic vector as:

$$\bar{\omega}^k \equiv \bar{\gamma}^{k+1} - \bar{\gamma}^k = \bar{\chi}^k - \bar{\chi}^{k+1}.$$

Notice that, elementwise

$$\bar{\omega}^k \begin{cases} > 0 & for\ event\ disabling \\ \leq 0 & for\ event\ re-enabling \end{cases}$$

The performance increment at iteration $k$ is given by:

$$\begin{aligned}\bar{\eta}^{k+1}-\bar{\eta}^{k} &= \left[I-\Pi^{k+1}\right]^{-1}\bar{\chi}^{k+1}-\left[I-\Pi^{k}\right]^{-1}\bar{\chi}^{k}\\ &= \left[I-\Pi^{k+1}\right]^{-1}\left[\bar{\chi}^{k}-\bar{\omega}^{k}\right]-\left[I-\Pi^{k}\right]^{-1}\bar{\chi}^{k}\\ &= \left(\left[I-\Pi^{k+1}\right]^{-1}-\left[I-\Pi^{k}\right]^{-1}\right)\bar{\chi}^{k}-\left[I-\Pi^{k+1}\right]^{-1}\bar{\omega}^{k}\\ &= \left[I-\Pi^{k+1}\right]^{-1}\left[\Pi^{k+1}-\Pi^{k}\right]\left[I-\Pi^{k}\right]^{-1}\bar{\chi}^{k}-\left[I-\Pi^{k+1}\right]^{-1}\bar{\omega}^{k}\\ &= -\left\{\left[I-\Pi^{k+1}\right]^{-1}\left[\Pi^{k}-\Pi^{k+1}\right]\bar{\eta}^{k}+\left[I-\Pi^{k+1}\right]^{-1}\bar{\omega}^{k}\right\}\end{aligned}$$

At $k=0$**,** the state transition cost matrix changes from $\Pi^0$ to $\Pi^1$ as a result of disabling selected controllable events leading to states with sufficiently negative performance. Let us denote the $i^{th}$column of a matrix $A$ as $(A)_i$, $ij^{th}$element of a matrix $A$ as $(A)_{ij}$, and the $i^{th}$ element of a vector $v$ as $(v)_i$; and $\ell$ and $j$ satisfy the following conditions:

$\delta(q_\ell,\sigma_j)=q_p$ and $d_{\ell j}^{S^k}\neq d_{\ell j}^{S^{k+1}}$

Then $\Pi^1\leq\Pi^0$; $\omega_\ell^0=\sum_j c_{\ell j}\left\{\tilde{\Pi}^0-\tilde{\Pi}^1\right\}_{\ell j}$; and

$$\begin{aligned}(\bar{\eta}^1-\bar{\eta}^0)_i &= -\left([I-\Pi^1]^{-1}[\Pi^0-\Pi^1]\,\bar{\eta}^0-[I-\Pi^1]^{-1}\bar{\omega}^0\right)_i\\ &= -\sum_\ell\left([I-\Pi^1]^{-1}\right)_{i\ell}\left(\sum_p\left(\sum_j\left(\tilde{\pi}_{\ell j}\eta_p^0+c_{\ell j}\tilde{\pi}_{\ell j}\right)\right)\right)\\ &= -\sum_\ell\left([I-\Pi^1]^{-1}\right)_{i\ell}\left(\sum_p\left(\sum_j\tilde{\pi}_{\ell j}(\eta_p^0+c_{\ell j})\right)\right)\end{aligned}$$

Since $[I-\Pi^1]^{-1}\geq 0$ elementwise and event disabling requires $(\eta_p^0+c_{\ell j})<0$ for all admissible $\ell$, $j$ and $p$, it follows from the above equation that $\bar{\eta}^1-\bar{\eta}^0\geq 0$ elementwise.

Next, iterations $k\geq 1$ are considered, for which some of the events disabled at $k=0$ are (possibly) re-enabled.

$$\omega_\ell^k=-\sum_j c_{\ell j}\left(\tilde{\Pi}^{k+1}-\tilde{\Pi}^k\right)_{\ell j}$$

$$\begin{aligned}\left(\bar{\eta}^{k+1}-\bar{\eta}^k\right)_i &= \left(\left[I-\Pi^{k+1}\right]^{-1}\left[\Pi^{k+1}-\Pi^k\right]\bar{\eta}^k-\left[I-\Pi^{k+1}\right]^{-1}\bar{\omega}^k\right)_i\\ &= \sum_\ell\left(\left[I-\Pi^{k+1}\right]^{-1}\right)_{i\ell}\left(\sum_p\left(\sum_j\tilde{\pi}_{\ell j}(\eta_p^k+c_{\ell j})\right)\right)\end{aligned}$$

Since $\left[I-\Pi^k\right]^{-1}\geq 0$ elementwise and event re-enabling requires $(\eta_p^k+c_{\ell j})\ \geq 0$ for all admissible $\ell,j$ and $p$, it follows from the above equations that $\bar{\eta}^{k+1}-\bar{\eta}^k\geq 0$ for $k\geq 0$.

**Proof of Proposition 4.2**: The optimal supervisor $S$ is synthesized by disabling and re-enabling certain controllable events at selected states. It is to be shown that the performance of any (controllable) supervisor $\tilde{S}$ is not superior to that of $S$, i.e., $\bar{\eta}^s\geq\bar{\eta}^{\tilde{s}}\quad\forall\tilde{S}\in\mathcal{S}$.

Let an arbitrary supervisor $\tilde{S}\in\mathcal{S}$ disable controllable events $\sigma_j$ at selected states $q_\ell$, which are not disabled by $S$, i.e., $(\eta_p^s+c_{\ell j})\ \geq 0$ with $\delta(q_\ell,\sigma_j)=q_p$, and enable some other controllable events $\sigma_{\tilde{j}}$ at selected states $q_\ell$, leading to state $q_{\tilde{p}}$, which are disabled by $S$, i.e., $(\eta_{\tilde{p}}^s+c_{\ell\tilde{j}})<0$ with $\delta(q_\ell,\sigma_{\tilde{j}})=q_{\tilde{p}}$ where $\ell$, $j$ and $\tilde{j}$ satisfy the condition $d_{\ell j}^S\neq d_{\ell j}^{\tilde{S}}$ and $d_{\ell\tilde{j}}^S\neq d_{\ell\tilde{j}}^{\tilde{S}}$.

Denoting the difference in event disabling characteristic vectors and the state transition cost matrices of $S$ and $\tilde{S}$ as: $\bar{\omega}\equiv\bar{\gamma}^{\tilde{s}}-\bar{\gamma}^s=\bar{\chi}^s-\bar{\chi}^{\tilde{s}}$, the corresponding difference in performance vectors is obtained as:

$$\begin{aligned}
\bar{\eta}^S - \bar{\eta}^{\tilde{S}} &= \left[I - \Pi^S\right]^{-1} \bar{\chi}^S - \left[I - \Pi^{\tilde{S}}\right]^{-1} \bar{\chi}^{\tilde{S}} \\
&= \left[I - \Pi^S\right]^{-1} \bar{\chi}^S - \left[I - \Pi^{\tilde{S}}\right]^{-1} \left[\bar{\chi}^S - \bar{\omega}\right] \\
&= \left(\left[I - \Pi^S\right]^{-1} - \left[I - \Pi^{\tilde{S}}\right]^{-1}\right) \bar{\chi}^S + \left[I - \Pi^{\tilde{S}}\right]^{-1} \bar{\omega} \\
&= \left[I - \Pi^{\tilde{S}}\right]^{-1} \left[\Pi^S - \Pi^{\tilde{S}}\right] \left[I - \Pi^S\right]^{-1} \bar{\chi}^S + \left[I - \Pi^{\tilde{S}}\right]^{-1} \bar{\omega} \\
&= \left[I - \Pi^{\tilde{S}}\right]^{-1} \left[\Pi^S - \Pi^{\tilde{S}}\right] \bar{\eta}^S + \left[I - \Pi^{\tilde{S}}\right]^{-1} \bar{\omega}
\end{aligned}$$

Letting $\Delta \equiv \Pi^S - \Pi^{\tilde{S}}$, the following equality conditions are defined:
$\Delta_{\ell p} = \sum_j \tilde{\pi}_{\ell j}$ and $\Delta_{\ell \tilde{p}} = -\sum_{\tilde{j}} \tilde{\pi}_{\ell \tilde{j}}$

Noting that the subscript $p$ depends on both $\ell$ and $j$, and the subscript $\tilde{p}$ depends on both $\tilde{\ell}$ and $\tilde{j}$, the product of the matrix $\Delta \in \Re^{n \times n}$ and the performance vector $\bar{\eta}^s \in \Re^n$ is obtained as:

$$\begin{aligned}
(\Delta \cdot \bar{\eta}^s)_\ell &\equiv \sum_p \Delta_{\ell p} \eta_p^s + \sum_{\tilde{p}} \Delta_{\ell \tilde{p}} \eta_{\tilde{p}}^s \\
&= \sum_p \left(\sum_j \tilde{\pi}_{\ell j}\, \eta_p^s\right) - \sum_{\tilde{p}} \left(\sum_{\tilde{j}} \tilde{\pi}_{\ell \tilde{j}}\, \eta_{\tilde{p}}^s\right)
\end{aligned}$$

The changes in the event disabling characteristic vector and the performance vector are then respectively expressed as follows:

$$\begin{aligned}
(\bar{\omega})_\ell &= \sum_i \tilde{\pi}_{\ell i} c_{\ell i} \\
&= \sum_p \left(\sum_j \tilde{\pi}_{\ell j} c_{\ell j}\right) - \sum_{\tilde{p}} \left(\sum_{\tilde{j}} \tilde{\pi}_{\ell \tilde{j}} c_{\ell \tilde{j}}\right)
\end{aligned}$$

$$\begin{aligned}
\left(\bar{\eta}^s - \bar{\eta}^{\tilde{s}}\right)_i &= \left(\left[I - \Pi^{\tilde{S}}\right]^{-1} (\Delta \cdot \bar{\eta}^s + \bar{\omega})\right)_i \\
&= \sum_\ell \left(\left[I - \Pi^{\tilde{S}}\right]^{-1}\right)_{i\ell} (\Delta \cdot \bar{\eta}^s + \bar{\omega})_\ell
\end{aligned}$$

The $\ell^{th}$ element of the vector $(\Delta \cdot \bar{\eta}^s + \bar{\omega})$ is obtained as:

$$\begin{aligned}
&(\Delta \bar{\eta}^s + \bar{\omega})_\ell \\
&= \sum_p \left(\sum_j \tilde{\pi}_{\ell j}\, \eta_p^s\right) - \sum_{\tilde{p}} \left(\sum_{\tilde{j}} \tilde{\pi}_{\ell \tilde{j}}\, \eta_{\tilde{p}}^s\right) \\
&\qquad + \sum_p \left(\sum_j \tilde{\pi}_{\ell j} c_{\ell j}\right) - \sum_{\tilde{p}} \left(\sum_{\tilde{j}} \tilde{\pi}_{\ell \tilde{j}} c_{\ell \tilde{j}}\right) \\
&= \sum_p \left(\sum_j \tilde{\pi}_{\ell j} \left(\eta_p^s + c_{\ell j}\right)\right) - \sum_{\tilde{p}} \left(\sum_{\tilde{j}} \tilde{\pi}_{\ell \tilde{j}} \left(\eta_{\tilde{p}}^s + c_{\ell \tilde{j}}\right)\right) \geq 0
\end{aligned}$$

because $(\eta_p^s + c_{\ell j}) \geq 0$ and $(\eta_{\tilde{p}}^s + c_{\ell \tilde{j}}) < 0$. Therefore, since $\left[I - \Pi^{\tilde{S}}\right]^{-1} \geq 0$ and $(\Delta\, \bar{\eta}^s + \bar{\omega}) \geq 0$ elementwise, it follows that $\left(\bar{\eta}^s - \bar{\eta}^{\tilde{s}}\right) \geq 0$ elementwise.

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