Introductory Materials
for ME 550.

[Taken from Kreyszig (1978)]

APPENDIX 1

SOME MATERIAL FOR REVIEW AND REFERENCE

A1.1 Sets

Sets are denoted by single capital letters $A, B, M, \cdots$ or by the use of braces, for example

$\{a, b, c\}$ denotes the set having the letters a, b, c as elements
$\{t \mid f(t) = 0\}$ denotes the set of all t at which the function f is zero.

Some symbols used in set theory are

$\varnothing$	Empty set (set which has no elements)
$a \in A$	a is an element of A
$b \notin A$	b is not an element of A
$A = B$	A and B are equal (are identical, consist of the same elements)
$A \neq B$	A and B are different (not equal)
$A \subset B$	A is a subset of B (each element of A also belongs to B). This is also written $B \supset A$.
$A \subset B, A \neq B$	A is a proper subset of B (A is a subset of B and B has at least one element which is not in A)
$A \cup B$	$=\{x \mid x \in A \text{ or } x \in B\}$ Union of A and B. See Fig. 77.
$A \cap B$	$=\{x \mid x \in A \text{ and } x \in B\}$ Intersection of A and B. See Fig. 77.
$A \cap B = \varnothing$	A and B are disjoint sets (sets without common elements)
$A - B$	$=\{x \mid x \in A \text{ and } x \notin B\}$ Difference of A and B. (Here B may or may not be a subset of A.) See Fig. 78. (See also Fig. 79.)
A^{c}	$= X - A$ Complement of A in X (where $A \subset X$) (notation $C_X A$ if confusion concerning X seems possible). See Fig. 80.

The following formulas result directly from the definitions:

(1a) $A \cup A = A$ $\qquad A \cap A = A$

(1b) $A \cup B = B \cup A$ $\qquad A \cap B = B \cap A$

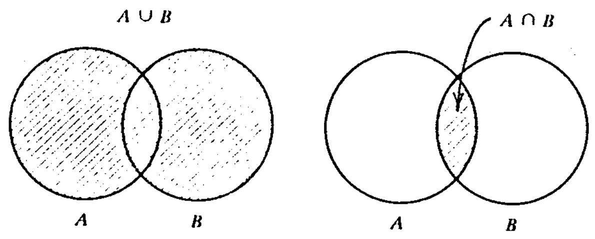
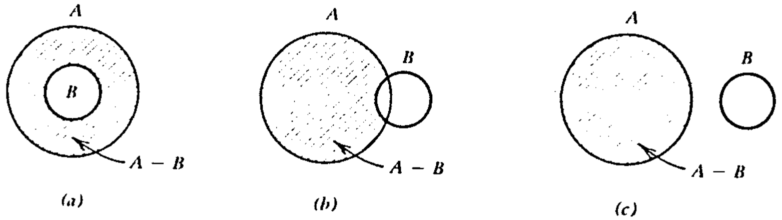


Fig. 77. Union $A\cup B$ (shaded) and intersection $A\cap B$ (shaded) of two sets A and B



(a) (b) (c)

Fig. 78. Difference $A-B$ (shaded) of two sets A (large disk) and B (small disk) if (a) $B\subset A$, (b) $A\cap B\neq\varnothing$ and $B\not\subset A$, (c) $A\cap B=\varnothing$

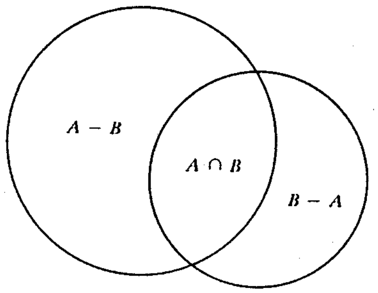


Fig. 79. Differences $A-B$ and $B-A$ and intersection $A\cap B$ of two sets A (large disk) and B (small disk)

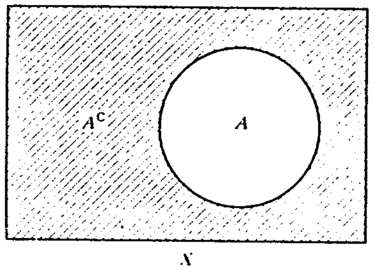


Fig. 80. Complement $A^c=X-A$ (shaded) of a subset A of a set X

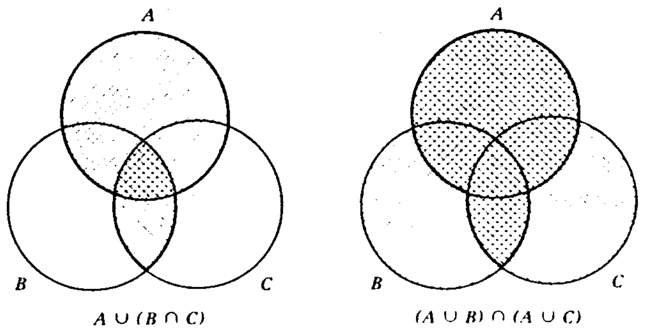


Fig. 81. Formula (1e)

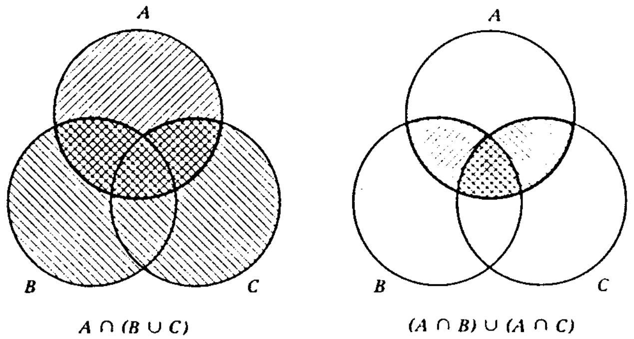


Fig. 82. Formula (1f)

(1c) $A\cup(B\cup C)=(A\cup B)\cup C,$ written $A\cup B\cup C$

(1d) $A\cap(B\cap C)=(A\cap B)\cap C$ written $A\cap B\cap C$

(1e) $A\cup(B\cap C)=(A\cup B)\cap(A\cup C)$

(1f) $A\cap(B\cup C)=(A\cap B)\cup(A\cap C)$

(1g) $A\cap B\subset A$ $A\cap B\subset B$

(1h) $A\cup B\supset A$ $A\cup B\supset B$

Furthermore,

$$\begin{array}{lccc} & A\subset B & \Longleftrightarrow\ A\cup B=B & \Longleftrightarrow\ A\cap B=A \\ (2) & A\subset C \text{ and } B\subset C & \Longleftrightarrow & A\cup B\subset C \\ & C\subset A \text{ and } C\subset B & \Longleftrightarrow & C\subset A\cap B. \end{array}$$

From the definition of a complement,

$$(A^{\mathrm{C}})^{\mathrm{C}} = A, \qquad X^{\mathrm{C}} = \varnothing, \qquad \varnothing^{\mathrm{C}} = X. \tag{3}$$

De Morgan's laws are (A and B any subsets of X)

$$\begin{aligned} (A \cup B)^{\mathrm{C}} &= A^{\mathrm{C}} \cap B^{\mathrm{C}} \\ (A \cap B)^{\mathrm{C}} &= A^{\mathrm{C}} \cup B^{\mathrm{C}} \end{aligned} \tag{4}$$

Obviously,

$$\begin{array}{ccccc} A \subset B & \Longleftrightarrow & A^{\mathrm{C}} \supset B^{\mathrm{C}} & & \\ A \cap B = \varnothing & \Longleftrightarrow & A \subset B^{\mathrm{C}} & \Longleftrightarrow & B \subset A^{\mathrm{C}} \\ A \cup B = X & \Longleftrightarrow & A^{\mathrm{C}} \subset B & \Longleftrightarrow & B^{\mathrm{C}} \subset A. \end{array} \tag{5}$$

The set of all subsets of a given set S is called the **power set** of S and is denoted by $\mathcal{P}(S)$.

The **Cartesian product** (or *product*) $X \times Y$ of two given nonempty sets X and Y is the set of all ordered pairs (x, y) with $x \in X$ and $y \in Y$. See Fig. 83.

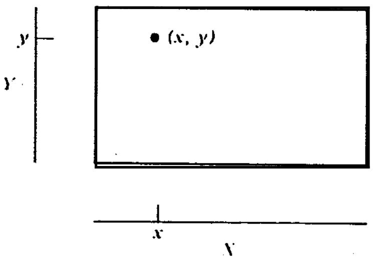


Fig. 83. A way of visualizing the Cartesian product $X \times Y$ of two sets X and Y

A set M is said to be **countable** if M is *finite* (has finitely many elements) or if we can associate positive integers with the elements of M so that to each element of M there corresponds a unique positive integer and, conversely, to each positive integer $1, 2, 3, \cdots$ there corresponds a unique element of M.

A1.2 Mappings

Let X and Y be sets and $A \subset X$ any subset. A **mapping** (or *transformation, functional relation, abstract function*) T from A into Y is obtained by associating with each $x \in A$ a single $y \in Y$, written $y = Tx$ and called the **image** *of x with respect to T*. The set A is called the *domain of definition* of T or, more briefly, the **domain** of T and is denoted by $\mathfrak{D}(T)$, and we write

$$\begin{aligned} T\colon \mathfrak{D}(T) &\longrightarrow Y \\ x &\longmapsto Tx. \end{aligned}$$

The **range** $\mathfrak{R}(T)$ of T is the set of all images; thus

$$\mathfrak{R}(T) = \{y \in Y \mid y = Tx \text{ for some } x \in \mathfrak{D}(T)\}.$$

The *image* $T(M)$ of any subset $M \subset \mathfrak{D}(T)$ is the set of all images Tx with $x \in M$. Note that $T(\mathfrak{D}(T)) = \mathfrak{R}(T)$.

An illustration of the situation is given in Fig. 84.

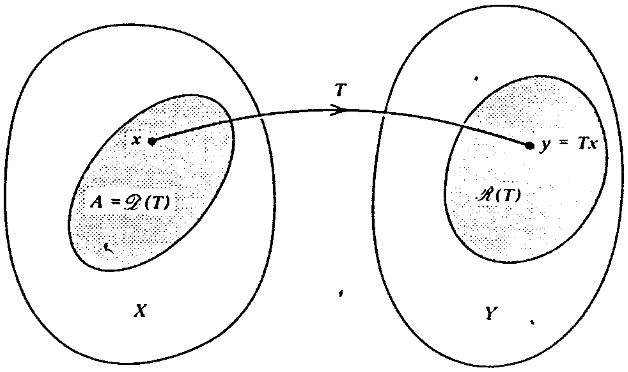


Fig. 84. Visualization of a mapping

The **inverse image** of a $y_0 \in Y$ is the set of all $x \in \mathfrak{D}(T)$ such that $Tx = y_0$. Similarly, the *inverse image* of a subset $Z \subset Y$ is the set of all $x \in \mathfrak{D}(T)$ such that $Tx \in Z$. Note that the inverse image of a $y_0 \in Y$ may be empty, a single point, or any subset of $\mathfrak{D}(T)$; this depends on y_0 and T.

A mapping T is **injective**, an **injection**, or **one-to-one** if for every $x_1, x_2 \in \mathscr{D}(T)$,

$$x_1 \neq x_2 \qquad \text{implies} \qquad Tx_1 \neq Tx_2;$$

that is, different points in $\mathfrak{D}(T)$ have different images, so that the inverse image of any point in $\mathfrak{R}(T)$ is a single point. See Fig. 85.

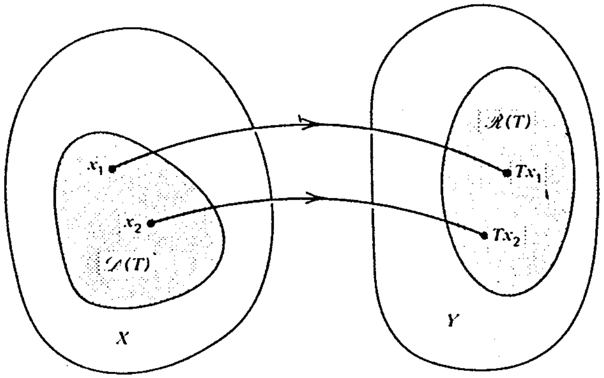


Fig. 85. Notation in connection with an injective mapping

T: $\mathfrak{D}(T) \longrightarrow Y$ is **surjective**, a **surjection**, or a mapping of $\mathfrak{D}(T)$ **onto** Y if $\mathfrak{R}(T) = Y$. See Fig. 86. Clearly,

$$\begin{aligned} \mathscr{D}(T) &\longrightarrow \mathscr{R}(T) \\ x &\longmapsto Tx \end{aligned}$$

is always surjective.

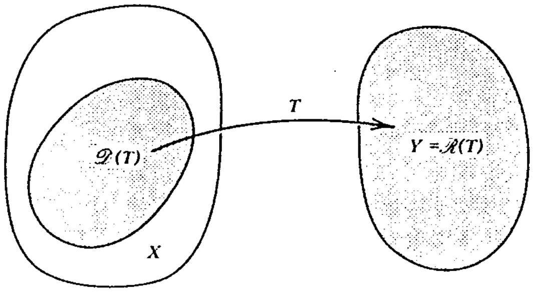


Fig. 86. Surjective mapping

T is **bijective** or a **bijection** if T is both injective and surjective. Then the **inverse mapping** T^{-1} of T: $\mathfrak{D}(T) \longrightarrow Y$ is the mapping T^{-1}: $Y \longrightarrow \mathfrak{D}(T)$ defined by $Tx_0 \longmapsto x_0$, that is, T^{-1} associates with each $y_0 \in Y$ that $x_0 \in \mathfrak{D}(T)$ for which $Tx_0 = y_0$. See Fig. 87.

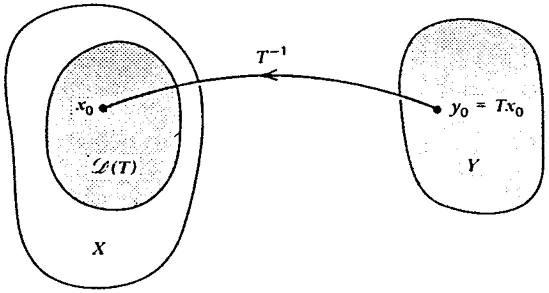


Fig. 87. Inverse $T^{-1}: Y \longrightarrow \mathfrak{D}(T) \subset X$ of a bijective mapping T

For an injective mapping T: $\mathfrak{D}(T) \longrightarrow Y$ the **inverse mapping** T^{-1} is defined to be the mapping $\mathfrak{R}(T) \longrightarrow \mathfrak{D}(T)$ such that $y_0 \in \mathfrak{R}(T)$ is mapped onto that $x_0 \in \mathfrak{D}(T)$ for which $Tx_0 = y_0$. See Fig. 88. Thus in this slightly more general use of the term "inverse" it is not required that T be a mapping *onto* Y; this convenient terminology employed by many authors is unlikely to cause misunderstandings in the present context.

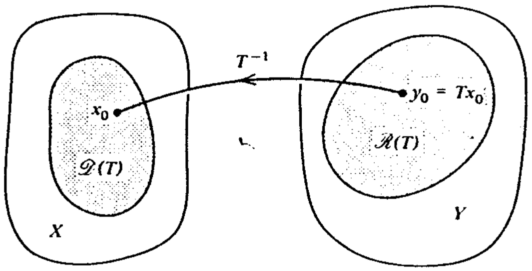


Fig. 88. Inverse $T^{-1}: \mathfrak{R}(T) \longrightarrow \mathfrak{D}(T)$ of an injective mapping T

Two mappings T_1 and T_2 are said to be **equal** if $\mathfrak{D}(T_1) = \mathfrak{D}(T_2)$ and $T_1 x = T_2 x$ for all $x \in \mathfrak{D}(T_1) = \mathfrak{D}(T_2)$.

The **restriction** $T|_B$ of a mapping T: $\mathscr{D}(T) \longrightarrow Y$ to a subset $B \subset \mathfrak{D}(T)$ is the mapping $B \longrightarrow Y$ obtained from T by restricting x to

B (instead of letting it vary in the whole domain $\mathfrak{D}(T)$); that is, $T|_B\colon B \longrightarrow Y$, $T|_B x = Tx$ for all $x \in B$. See Fig. 89.

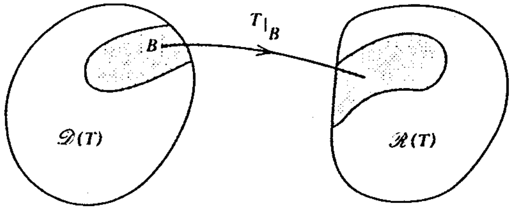


Fig. 89. Restriction $T|_B$ of a mapping T to a subset $B \subset \mathfrak{D}(T)$

An **extension** of T from $\mathfrak{D}(T)$ to a set $C \supset \mathfrak{D}(T)$ is a mapping $\tilde{T}$ such that $\tilde{T}|_{\mathfrak{D}(T)} = T$, that is, $\tilde{T}x = Tx$ for all $x \in \mathfrak{D}(T)$.

An extension $\tilde{T}$ of T is said to be *proper* if $\mathfrak{D}(T)$ is a proper subset of $\mathfrak{D}(\tilde{T})$; thus $\mathfrak{D}(\tilde{T}) - \mathfrak{D}(T) \neq \varnothing$, that is, $x \in \mathfrak{D}(\tilde{T})$ for some $x \notin \mathfrak{D}(T)$.

Composition of mappings is defined and denoted as follows. If $T\colon X \longrightarrow Y$ and $U\colon Y \longrightarrow Z$, then

$$x \longmapsto U(Tx) \qquad (x \in X)$$

defines a mapping of X into Z which is written $U \circ T$ or simply UT, thus

$$UT\colon X \longrightarrow Z, \qquad x \longmapsto UTx \qquad (x \in X),$$

and is called the **composite** or **product** of U and T. See Fig. 90. Note that T is applied first and the order is essential: TU would not even

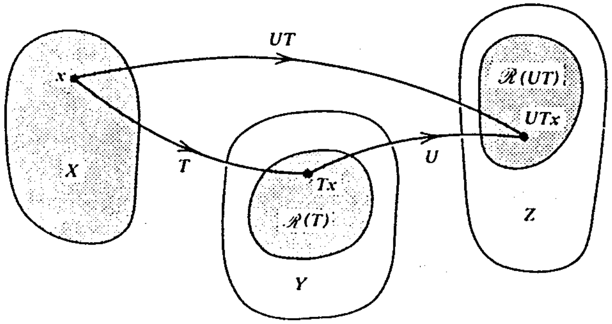


Fig. 90. Composition of two mappings

make sense, in general. If $T\colon X \longrightarrow Y$ and $U\colon Y \longrightarrow X$, both $UT\colon X \longrightarrow X$ and $TU\colon Y \longrightarrow Y$ make sense but are different if $X \neq Y$. (Even if $X = Y$, those two mappings will be different, in general.)

A1.3 Families

A **sequence** (x_n) of real or complex numbers is obtained if we associate with each positive integer n a real or complex number x_n. This process can be regarded as a mapping of $\mathbf{N} = \{1, 2, \cdots\}$ into the real or complex numbers, x_n being the image of n. The set $\mathbf{N}$ is called the **index set** of the sequence.

This process of "indexing" can be generalized. Instead of $\mathbf{N}$ we may take any nonempty set I (finite, countable or uncountable) and map I into any other given nonempty set X. This gives a **family of elements** of X, written $(x_\alpha)_{\alpha \in I}$ or simply (x_α), where $x_\alpha \in X$ is the image of $\alpha \in I$. Note that it may happen that $x_\alpha = x_\beta$ for some $\alpha \neq \beta$ in I. The set I is called the **index set** of the family. A **subfamily** of a family is obtained if we restrict the indexing mapping to a nonempty subset of the index set.

If the elements of X are subsets of a given set, we obtain a **family of subsets** $(B_\alpha)_{\alpha \in I}$ where B_α is the image of α.

The **union** $\bigcup\limits_{\alpha \in I} B_\alpha$ of the family (B_α) is the set of elements each of which belongs to at least one B_α, and the **intersection** $\bigcap\limits_{\alpha \in I} B_\alpha$ is the set of elements which belong to every B_α, $\alpha \in I$. If $I = \mathbf{N}$, we write

$$\bigcup_{\alpha=1}^{\infty} B_\alpha \qquad \text{and} \qquad \bigcap_{\alpha=1}^{\infty} B_\alpha;$$

and if $I = \{1, 2\}$, we write $B_1 \cup B_2$ and $B_1 \cap B_2$, respectively.

One must carefully distinguish a *family* $(x_\alpha)_{\alpha \in I}$ from the *subset* of X whose elements are the elements of the family, which is the image of the index set I under the indexing mapping.

To any nonempty subset $M \subset X$ we can always find a family of elements of X the set of whose elements is M. For instance, we may take the family defined by the *natural injection* of M into X, that is, the restriction to M of the identity mapping $x \longmapsto x$ on X.

A1.4 Equivalence Relations

Let X and Y be given nonempty sets. Any subset R of the Cartesian product $X \times Y$ (see before) is called a (*binary*) **relation.** $(x, y) \in R$ is also written $R(x, y)$.

An **equivalence relation** *on* X is a relation $R \subset X \times X$ such that

$$
(1) \qquad \begin{array}{rlll}
R(x,x) & \text{for all } x \in X & & (\textit{Reflexivity}) \\
R(x,y) & \text{implies} & R(y,x) & (\textit{Symmetry}) \\
R(x,y) \text{ and } R(y,z) & \text{implies} & R(x,z) & (\textit{Transitivity})
\end{array}
$$

When R is an equivalence relation on X, then $R(x, y)$ is usually written $x \sim y$, (read "*x is equivalent to y*"). In this case, (1) becomes

$$
\begin{array}{rcl}
 & x \sim x & \\
x \sim y & \Longrightarrow & y \sim x \\
x \sim y \text{ and } y \sim z & \Longrightarrow & x \sim z.
\end{array}
$$

The *equivalence class* of any $x_0 \in X$ is the set of all $y \in X$ which are equivalent to x_0, and any such y is called a *representative* of the class. The equivalence classes with respect to R constitute a *partition* of X.

By definition, a **partition** of a nonempty set X is a family of nonempty subsets of X which are pairwise disjoint and whose union is X.

A1.5 Compactness

A **cover** (or *covering*) of a subset M of a set X is a family of subsets of X, say, $(B_\alpha)_{\alpha \in I}$ (I the index set), such that

$$M \subset \bigcup_{\alpha \in I} B_\alpha.$$

In particular, if (B_α) is a cover of X, then

$$\bigcup_{\alpha \in I} B_\alpha = X.$$

A cover is said to be **finite** if it consists of only finitely many sets B_α. If $X = (X, \mathcal{T})$ is a topological space (for instance, a metric space; cf. Sec. 1.3), that cover is said to be **open** if all the B_α's are open sets.

A topological space $X = (X, \mathcal{T})$ is said to be

(a) **compact** if every open cover of X contains a finite cover of X, that is, a finite subfamily which is a cover of X.

(b) **countably compact** if every countable open cover of X contains a finite cover of X,

(c) **sequentially compact** if every sequence in X contains a convergent subsequence.

A subset $M \subset (X, \mathcal{T})$ is said to be *compact* (*countably compact, sequentially compact*) if M considered as a subspace $(M, \mathcal{T}_M)$ is compact (countably compact, sequentially compact, respectively); here the *induced topology* $\mathcal{T}_M$ on M consists of all sets $M \cap A$ with $A \in \mathcal{T}$.

For a metric space, the three concepts of compactness are equivalent, that is, one implies the others.

A1.6 Supremum and Infimum

A subset E of the real line $\mathbf{R}$ is **bounded above** if E has an **upper bound,** that is, if there is a $b \in \mathbf{R}$ such that $x \leqq b$ for all $x \in E$. Then if $E \neq \varnothing$, there exists the **supremum** of E (or *least upper bound* of E), written

$$\sup E,$$

that is, the upper bound of E such that $\sup E \leqq b$ for every upper bound b of E. Also

$$\sup C \leqq \sup E$$

for every nonempty subset $C \subset E$.

Similarly, E is **bounded below** if E has a **lower bound,** that is, if there is an $a \in \mathbf{R}$ such that $x \geqq a$ for all $x \in E$. Then if $E \neq \varnothing$, there exists the **infimum** of E (or *greatest lower bound* of E), written

$$\inf E,$$

that is, the lower bound of E such that $\inf E \geqq a$ for every lower bound a of E. Also

$$\inf C \geqq \inf E$$

for every nonempty subset $C \subset E$.

E is **bounded** if E is both bounded above and bounded below. Then if $E \neq \varnothing$,

$$\inf E \leqq \sup E.$$

If for a mapping $T\colon \mathcal{D}(T) \longrightarrow \mathbf{R}$ the range $\mathcal{R}(T)$ (assumed nonempty) is bounded above, its supremum is denoted by

$$\sup_{x \in \mathcal{D}(T)} Tx,$$

and if $\mathcal{R}(T)$ is bounded below, its infimum is denoted by

$$\inf_{x \in \mathcal{D}(T)} Tx.$$

Similar notations are used in connection with subsets of $\mathcal{R}(T)$.

A1.7 Cauchy Convergence Criterion

A number a is called a *limit point* of a (real or complex) sequence of numbers (x_n) if for every given $\varepsilon > 0$ we have

$$|x_n - a| < \varepsilon \qquad \text{for infinitely many } n.$$

The *Bolzano-Weierstrass theorem* states that a *bounded* sequence (x_n) has at least one limit point. Here it is essential that a sequence has infinitely many terms, by definition.

A (real or complex) sequence (x_n) is said to be *convergent* if there is a number x such that, for every given $\varepsilon > 0$, the following condition holds:

$$|x_n - x| < \varepsilon \qquad \text{for all but finitely many } n.$$

This x is called the *limit* of the sequence (x_n).

The limit of a convergent sequence is unique. Note that it is a limit point (why?) and is the only limit point which a convergent sequence has.

We state and prove the Cauchy convergence theorem, whose importance is due to the fact that for deciding about convergence one need not know the limit.

Cauchy Convergence Theorem. *A (real or complex) sequence (x_n) is convergent if and only if for every $\varepsilon > 0$ there is an N such that*

$$(1) \qquad |x_m - x_n| < \varepsilon \qquad \textit{for all } m, n > N.$$

Proof. (*a*) If (x_n) converges and c is its limit, then for every given $\varepsilon > 0$ there is an N (depending on ε) such that

$$|x_n - c| < \frac{\varepsilon}{2} \qquad \text{for every } n > N,$$

so that by the triangle inequality for $m, n > N$ we obtain

$$|x_m - x_n| \leqq |x_m - c| + |c - x_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

(*b*) Conversely, suppose that the statement involving (1) holds. Given $\varepsilon > 0$, we can choose an $n = k > N$ in (1) and see that every x_m with $m > N$ lies in the disk D of radius ε about x_k. Since there is a disk which contains D as well as the finitely many $x_n \notin D$, the sequence (x_n) is bounded. By the Bolzano-Weierstrass theorem it has a limit point a. Since (1) holds for every $\varepsilon > 0$, an $\varepsilon > 0$ being given, there is an N^* such that $|x_m - x_n| < \varepsilon/2$ for $m, n > N^*$. Choosing a fixed $n > N^*$ such that $|x_n - a| < \varepsilon/2$, by the triangle inequality we have for all $m > N^*$

$$|x_m - a| \leqq |x_m - x_n| + |x_n - a| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

which shows that (x_m) is convergent with the limit a. ∎