

ME 550. FOUNDATIONS OF ENGINEERING SYSTEMS ANALYSIS

Appendix #02: Equivalence Relations, Partitions, and Functions

Definition A2-1: A family of nonempty subsets of a nonempty set, represented as $\{A_\alpha : \alpha \in \mathbf{I}\}$ where \mathbf{I} is the index set, is called a partition if $\bigcup_{\alpha \in \mathbf{I}} A_\alpha = A$ and $A_\alpha \cap A_\beta = \emptyset \forall \alpha \neq \beta$. That is, a partition is a family of subsets that are exhaustive and pairwise disjoint.

Example A2-1: Let $\mathbf{Z} = \{0, 1, 2, 3, \dots\}$ and the index set $\mathbf{I} = \{0, 1, 2, \dots, K-1\}$. Define, for $i \in \mathbf{I}$, $Z_{/i} = \{i, K+i, 2K+i, 3K+i, \dots\}$. Then, $\{Z_{/i} : i \in \mathbf{I}\}$ is a partition of \mathbf{Z} .

Example A2-2: Let $A = C[0, T]$ be the set of all real-valued continuous functions defined on the interval $[0, T]$. Let $\mathbf{I} = [0, \infty)$ and $A_\alpha = \left\{ \varphi \in C[0, T] : \int_0^T |\varphi(t)|^2 dt = |\alpha|^2 < \infty \right\}$. Then, $A_\alpha \cap A_\beta = \emptyset \forall \alpha \neq \beta$ and also check that $\bigcup_{\alpha \in \mathbf{I}} A_\alpha = A$ by using the fact that $\int_0^T |\varphi(t)|^2 dt < \infty \forall \varphi \in A$.

Definition A2-2: Let S be a nonempty set. A relation \mathfrak{R} on S is defined as any subset of the Cartesian product $S \times S$. That is, the ordered pair $(x, y) \in S \times S$ belongs to \mathfrak{R} , then we say $x \mathfrak{R} y \Leftrightarrow x$ is related to y under \mathfrak{R} .

Definition A2-3: A relation \mathfrak{R} is reflexive on a set S if $x \mathfrak{R} x \forall x \in S$.

Definition A2-4: A relation \mathfrak{R} is symmetric on a set S if $x \mathfrak{R} y \Leftrightarrow y \mathfrak{R} x \forall x, y \in S$.
A relation \mathfrak{R} is asymmetric on a set S if $(x \mathfrak{R} y \text{ and } y \mathfrak{R} x) \Leftrightarrow x = y$.

Definition A2-5: A relation \mathfrak{R} is transitive on a set S if $((x \mathfrak{R} y \text{ and } y \mathfrak{R} z) \Rightarrow x \mathfrak{R} z) \forall x, y, z \in S$.

Definition A2-6: A relation \mathfrak{R} on a set S is said to be an equivalence relation if it is reflexive, symmetric and transitive. The statement x equivalent to y is often abbreviated as $x \sim y$.

Example A2-3: The relation $<$ on the real line is not reflexive and not symmetric; but it is transitive. The relation \leq is reflexive and transitive but it is not symmetric.

Example A2-4: Let $A = C[0, T]$ be the set of all real-valued continuous functions defined on the interval $[0, T]$ and the relation \mathfrak{R} on A be defined as: $x \mathfrak{R} y \Leftrightarrow \int_0^T |x(t)|^2 dt = \int_0^T |y(t)|^2 dt$. Then, \mathfrak{R} is an equivalence relation even though $|x(t)|^2 \neq |y(t)|^2$ is possible at some points in $[0, T]$.

Definition A2-7: If \mathfrak{R} is an equivalence relation on S and $x \in S$ the equivalence class determined by x is the subset $C_x = \{y \in P : y \sim x\}$.

Example A2-5: Let us continue with Example A2-4, $C_x = \left\{ y \in C[0, T] : \int_0^T |y(t)|^2 dt = \int_0^T |x(t)|^2 dt \right\}$. As a specific example, consider the root mean square (rms) value of signals $x(t) = \text{Sin}\left(\frac{2\pi t}{T}\right)$. Then $y(t) = \text{Sin}\left(\frac{2\pi kt}{T} + \varphi\right)$ belongs to the above equivalence class C_x where k is any positive integer and $\varphi \in [-\pi, \pi)$.

Theorem A2-1: Let \mathfrak{R} be an equivalence relation on a set S . Then, the family of all equivalence classes is a partition of S .

Proof: Let C_x and C_y be two equivalence classes on S . We want to show that either $C_x \cap C_y = \emptyset$ or $C_x = C_y$. Suppose $C_x \cap C_y \neq \emptyset$. Let $z \in C_x \cap C_y$. Then, by definition, $z \mathfrak{R} x$ and $z \mathfrak{R} y$. Let $\xi \in C_x$, i.e., $\xi \mathfrak{R} x$. Then, by applying the properties of symmetry and transitivity, we have:

$$\xi \mathfrak{R} x, z \mathfrak{R} x, z \mathfrak{R} y \Rightarrow \xi \mathfrak{R} x, x \mathfrak{R} z, z \mathfrak{R} y \Rightarrow \xi \mathfrak{R} y \Rightarrow \xi \in C_y \Rightarrow C_x \subset C_y$$

By a similar argument, we can show that $C_y \subset C_x$. Hence, $C_x = C_y$. Then, the only possibilities are: $C_x \cap C_y = \emptyset$ or $C_x = C_y$. Since \mathfrak{R} is reflexive, we have $x \in C_x \forall x \in S$. It follows then $\bigcup_{x \in S} C_x = S$. ♦

Theorem A2-2: Let $\{S_\alpha : \alpha \in \mathbf{I}\}$ be a partition of a set S . Then, the relation \mathfrak{R} , defined as $x \mathfrak{R} y \Leftrightarrow x, y \in S_\alpha$ for some $\alpha \in \mathbf{I}$, is an equivalence relation on S .

Proof: Clearly \mathfrak{R} is reflexive because $x \in P_\alpha$ for some $\alpha \in \mathbf{I}$. Since $\bigcup_{\alpha \in \mathbf{I}} P_\alpha = P$ and $P_\alpha \cap P_\beta = \emptyset \forall \alpha \neq \beta$, we argue that \mathfrak{R} is symmetric because x and y belong to the same P_α and also that \mathfrak{R} is transitive because of the same reason. ♦

Example A2-6: Let $L_2([a, b])$ denote the set of all complex square-integrable functions on $[a, b]$, i.e.,

$$L_2[a, b] = \left\{ x \in \mathbf{C} : \int_a^b |x(t)|^2 dt < \infty \right\}$$

Define the relation \mathfrak{R} by $x \mathfrak{R} y \Leftrightarrow \int_a^b |x(t) - y(t)|^2 dt = 0$, i.e., $x \sim y \Leftrightarrow \{t \in [a, b] : x(t) = y(t) \text{ almost everywhere}\}$.

Functions and Mappings

Definition A2-8: Let X and Y be two nonempty sets. A function f from X into Y , denoted as $f : X \rightarrow Y$, is a nonempty subset of the Cartesian product $X \times Y$ with the property that no two distinct members of f have the same first coordinate. That is, if $(x, y) \in f$ and $(x, z) \in f$, then $y = z$.

The domain of a function $f : X \rightarrow Y$, denoted as $D(f)$, is the subset of X that contains all first coordinates of the members of f . If $x \in D(f)$, then the second coordinate of f is called the value of f at x which is a unique member often denoted as $f(x)$. That is, $x \in D(f)$ iff $(x, y) \in f$ for some $y \in Y$; and $y \in Y$ is called the image of x under f .

The range of a function f from X into Y , denoted as $R(f)$, is the subset of Y that contains all second coordinates of the members of f . That is, $y \in R(f)$ iff $(x, y) \in f$ for some $x \in X$.

Definition A2-9: A function f is called the total function or mapping of X into Y , denoted as $f : X \rightarrow Y$ if $D(f) = X$. If is a proper subset of X , then f is called a partial function. In both cases, $R(f) \subset Y$.

Definition A2-10: A mapping $f : X \rightarrow Y$ is called onto (or surjective) $R(f) = Y$. And f is called one-to-one (or injective) if distinct points of X have distinct images. That is, if $(x_1, y) \in f$ and $(x_2, y) \in f$, then $x_1 = x_2$.

Definition A2-11: A mapping $f : X \rightarrow Y$ is called one-to-one correspondence (or bijective) if it is both one-to-one and onto.

Definition A2-12: The inverse image of a set $B \subset R(f)$ is denoted as $f^{-1}(B) = \{x \in D(f) : f(x) \in B\}$.

Definition A2-13: The composite $gf(\bullet)$ of two functions $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ exists iff $D(g) \subseteq R(f)$. Note that the existence of gf does not imply the existence of fg or vice versa.

Definition A2-14: The graph of a function $f : X \rightarrow Y$ is a subset of the Cartesian product $X \times Y$ made up of all ordered pairs (x, y) under the constraint $y = f(x)$. That is, $graph(x) = \{(x, y) \in X \times Y : y = f(x)\}$.

Example A2-7: Let $C^n([a, b])$ with $n \in \mathbb{Z}$ denote the set of all real-valued functions x defined on the interval $[a, b]$ such that the derivatives $\frac{d^k x}{dt^k}$ of order $k \leq n$ exist and are continuous on $[a, b]$. We define

$C^\infty[a, b] = \bigcap_{n=0}^{\infty} C^n[a, b]$, i.e., $x \in C^\infty[a, b]$ iff x has continuous derivatives of all orders. Let $P^n(D)$ denote the

differential operator defined as: $P^n[D] = \sum_{j=0}^n \alpha_j \frac{d^j}{dt^j}$ where α_j 's are constants. It follows that $P^n(D)$ can be

considered as a mapping of $C^n[a, b]$ into $C^0[a, b]$. The interval $[a, b]$ can be replaced by any arbitrary interval

including the real line \mathfrak{R} . For that matter, we could even replace it by any nonempty $\Omega \subset \mathfrak{R}^m$, in which $C^n(\Omega)$ is

the collection of all functions $u = u(x_1, x_2, \dots, x_m)$ defined on Ω with the property that all partial derivatives up to the order n are continuous; often it is said that the function u is C^n -continuous. Also, $C^\infty(\Omega) = \bigcap_{n=0}^{\infty} C^n(\Omega)$.

HW#A2-1: Verify the assertion in Example A2-2.

HW#A2-2: Verify the assertion in Example A2-4.

HW#A2-3: Verify the assertion in Example A2-5.

HW#A2-4: Going back to Example A2-1, we define the operation of modulo K on the set $\mathbf{Z} = \{0, 1, 2, 3, \dots\}$ such that $m \sim n \Leftrightarrow \text{Rem}(m/K) = \text{Rem}(n/K)$ where $\text{Rem}(n/K)$ is defined as the remainder of n divided by K . Having the index set $\mathbf{I} = \{0, 1, 2, \dots, K-1\}$, define $\mathbf{Z}_{/i} := \{i, K+i, 2K+i, 3K+i, \dots\}$ where $i \in \{0, 1, 2, \dots, K-1\}$. Show that

each $\mathbf{Z}_{/i}$, $i \in \{0, 1, 2, \dots, K-1\}$ is an equivalence class of elements and is a partitioning of \mathbf{Z} . (Note that $\mathbf{Z} = \bigcup_{i=0}^{K-1} \mathbf{Z}_{/i}$

and $\mathbf{Z}_{/i} \cap \mathbf{Z}_{/j} = \emptyset \forall i \neq j$.)

HW#A2-5: Let $C_\sigma = \{s \in \mathbf{C} : \text{Re}(s) = \sigma\}$. Let $L_{2,\sigma}(-i\infty, i\infty)$ be the set of all complex-valued functions $x(s)$ defined on C_σ such that $\frac{1}{2\pi i} \int_{C_\sigma} |x(s)|^2 ds < \infty$. Let $g(s)$ be a bounded continuous complex-valued function defined

on C_σ . Then, show that $y(s) = g(s)x(s) \forall s \in C_\sigma$ represents a mapping of $L_{2,\sigma}(-i\infty, i\infty)$ into itself.

HW#A2-6: Let $L_1((-\infty, \infty))$ be the set of all absolute-integrable functions whose domain is the set $(-\infty, \infty)$ of all real numbers. As an example, consider a resistance-capacitance (R-C) network in which the time constant

$\tau \text{ sec} = (R \text{ MegaOhm}) \times (C \text{ microsec})$. A model of the network is given as: $y(t) = \int_{-\infty}^t h(t-\varphi)u(\varphi)d\varphi$ where

$h(t) = \tau \exp(-t/\tau)$ is the impulse response of the network. Verify whether the model is an injective mapping of $L_1((-\infty, \infty))$ into $L_1((-\infty, \infty))$.