ME 550. FOUNDATIONS OF ENGINEERING SYSTEMS ANALYSIS

Appendix #03: Cardinality of Sets and Zorn's Lemma

The notion of cardinality of a set Ω, denoted as card(Ω), is related to the number of elements in Ω. If Ω has finitely many (say n) elements, then card(Ω)=n. However, the concept is not straightforward for infinite sets. We will consider the following three cases for two sets X and Y (regardless of whether they are finite or infinite): card(X)=card(Y); card(X)$\leq$card(Y); card(X)<card(Y).

Definition A3-1: Let X and Y be two nonempty sets. Then,

- card(X= card(Y) if there exists a bijective mapping between X and Y.
- card(X)$\leq$ card(Y) if there exists an injective mapping from X into Y.
- card(X)< card(Y) if every injective mapping from X into Y is NOT onto, i.e., its range is a strictly proper subset of Y.

Theorem A3-1: Let X and Y be two sets. If card(X)$\leq$card(Y) and card(Y)$\leq$card(X), then card(X)=card(Y).

Proof: It is obvious from Definition A3-1. ♦

Definition A3-2: A set is called *countable* if there exists a bijective mapping between X and the set, $\mathbf{N} = \{1,2,3,\cdots\}$, of natural numbers. A set which is either finite or countable is called *at most countable.* An infinite set which is not countable is called *uncountable.*

Remark A3-1: card(a finite set) $\leq$ card(an at most countable set) $\leq$ card(a countable set)=card(another countable set) < card(an uncountable set).

Remark A3-2: card($\varnothing$)=0.

Remark A3-3: The set, **J,** of all integers and the set, **N,** of all positive integers belong to the same class of cardinality even though **N** is a proper subset of **J**. This fact may appear to be counter-intuitive from the perspective of finite sets.

Example A3-1: To show that the set, **J**, of all integers is countable, i.e., card(**Z**)=card(**N**), we find a bijective mapping $f:\mathbf{N}\to\mathbf{J}$ as follows: $f(n)=\begin{cases}\frac{n}{2} \text{ for } n \text{ even}\\ -\frac{n-1}{2} \text{ for } n \text{ odd}\end{cases}$

Similarly, to show that the set, $\mathbf{Z}=\{0,1,2,\cdots\}$ of non-negative integers is countable, i.e., card(**Z**)=card(**N**), we find a bijective mapping $f:\mathbf{N}\to\mathbf{Z}$ as follows: $f(n)=n-1$.

Theorem A3-2: Let A be a countable set and B_n be the set of all n-tuples $(a_1,a_2,\cdots,a_n)$ where $a_k\in A$. Then, B_n is countable.

Proof: For $n=1$, $B_1=A$ is countable. Suppose B_{n-1} is countable for some $n>1$. Then, the elements of B_n are of the form (b,a) where $b\in B_{n-1};a\in A$. For every fixed b, the set of pairs (b,a) bears an equivalence relation with A, and hence is countable. Therefore, B_n is the union of a countable set of countable sets. By Theorem A3-2, B_n is countable. Now, the proof is completed by induction. ♦

Corollary to Theorem A3-2: The set, $Q \equiv \{\frac{m}{n} : m \in J;\ n \in J - \{0\}\}$ of rational numbers is countable. ♦

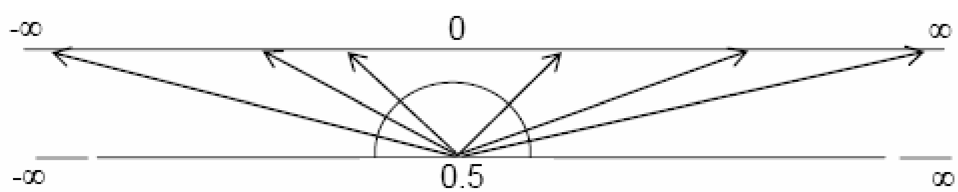
Theorem A3-3: Let $\{s_i\}$ be a sequence of countable sets, i.e., s_i is a countable set for every $i = 1,2,3,\cdots$. Then, a countable union of s_i's is also a countable set.
Proof: See Rudin (p. 29). ♦

Theorem A3-4: The set $(0,1) = \{x : 0 < x < 1\} \subset \mathbf{R} = (-\infty, \infty)$ is uncountable.

Proof: Let us assume that the set $(0,1)$ is countable, i.e., $(0,1) \sim N$. That is, we assume the existence of a bijective mapping between the sets $(0,1)$ and $\mathbf{N}$, in which every $x_i \in (0,1)$ corresponds to a unique $i \in \mathbf{N}$. Let us consider a real number $x_i \in (0,1)$ that is written as $0.\delta_{i1}\delta_{i2}\delta_{i3}\delta_{i4}\cdots$ where $0 \le \delta_{ij} \le 9$ is an integer. We define $\tilde{\delta}_{ii} = 9 - \delta_{ii}$ so that $0 \le \tilde{\delta}_{ii} \le 9$. Consider the real number $y = 0.\tilde{\delta}_{11}\tilde{\delta}_{22}\tilde{\delta}_{33}\tilde{\delta}_{44}\cdots$ that certainly belongs to the set (0,1). Since y must differ from any of the x_i's defined above, the bijective mapping from $\mathbf{N}$ onto (0,1) is impossible. Therefore, the above assumption that the set $(0,1)$ is countable is false. ♦

Remark A3-4: The sets $\mathbf{N}$ and $\mathbf{R}$ belong to the different classes of cardinality.

Remark A3-5: The sets $(0,1)$ and $\mathbf{R}$ have the same cardinality. This will be clear after we study the topological spaces. We will show that (0,1) and $\mathbf{R}$ are homeomorphic under the usual topology. Loosely speaking, this means that the sets (0,1) and $\Re$ are indistinguishable from the topological perspectives. Candidate mappings are: $f : (0,1) \to (-\infty, \infty)$ with $f(x) = \tan^{-1}\left(\left(x - \frac{1}{2}\right)\pi\right)$ and $f(x) = \frac{2x-1}{x(x-1)}$, which are both bijective and bicontinuous. Therefore, f is a homeomorphism. Let us look at the picture from a geometric point of view in the diagram below.



Represent $\mathbf{R}$ by an infinite straight line axis on which each point represents a unique real number. Now draw the following figure after bending the line segment (0,1) into a semicircle. If the lines are drawn from the center of the semicircle intersecting both the semicircle and the infinite straight line axis, the points of intersection can be paired as a bijection from (0,1) onto $\mathbf{R}$.

Zorn's Lemma

Definition A3-3: A relation $\le$ on a nonempty set S is said to be a partial ordering if

- $x \le x \quad \forall x \in S$
- $(x \le y \text{ and } y \le x) \Rightarrow x = y \quad \forall x, y \in S$
- $(x \le y \text{ and } y \le z) \Rightarrow x \le z \quad \forall x, y, z \in S$

A set S is said to be partially ordered if S has a defined partial ordering.

Definition A3-4: A partial ordering $\le$ on a nonempty set S is said to be a total ordering if, in addition,

- *either* $x \le y$ *or* $y \le x$ *for any two points* $x, y \in S$

A set S is said to be totally ordered if S has a defined total ordering.

Definition A3-5: Let $\le$ be a partial ordering on a nonempty set S and let $A \subset S$ be nonempty. Then, the set A is said to be a chain if A is totally ordered, i.e., either $x \le y$ or $y \le x$ for any two points $x, y \in A$.

Definition A3-6: Let $\le$ be a partial ordering on a nonempty set S and let $A \subset S$ be nonempty. Then, $\tilde{a} \in S$ is said to be an upper bound of A if $y \le \tilde{a} \quad \forall y \in A$. Furthermore, if $\tilde{a} \in A$, then $\tilde{a}$ is a maximal element of A.

Remark A3-6: A chain is a restriction of a partial ordering that yields a total ordering.

Example A3-2: The collection P of all open subsets of $\mathbf{R} \times \mathbf{R}$ is partially ordered but not totally ordered.

Example A3-3: The collection T of all open disks $\{(x^2 + y^2) < \rho^2\}$ in $\mathbf{R} \times \mathbf{R}$ is totally ordered. Furthermore, $\mathsf{T} \subset \mathsf{P}$ is a maximally totally ordered set, i.e., if any member of P which is not in T is adjoined with T, then the resulting collection of sets is no longer totally ordered by $\subset$.

Theorem A3-5 (Zorn's Lemma): If every chain in a partially ordered set S has an upper bound, then S has a maximal element.

Remark A3-7: Zorn's Lemma can be interpreted as follows: In a nonempty set c with partial ordering $\le$, let every chain $U \subset S$ have an upper bound, i.e., $\exists x \in S$ s.t. $x \ge \alpha \ \forall \alpha \in U$. Then, S has a maximal element. . (Equivalent to Zorn's Lemma.)

Theorem A3-7 (Axiom of Choice): Let $\boldsymbol{I}$ be an index set for a partially ordered set S and $\exists$ a nonempty $S_\alpha \subset S \ \forall \alpha \in \boldsymbol{I}$. Let Σ be collection of all such functions S_α. Then, one can define a function $\vartheta : \boldsymbol{I} \to \Sigma$.

Theorem A3-6 (Hausdorff Maximality Theorem): Every partially ordered set contains a maximally totally ordered set. (Equivalent to Zorn's Lemma.)

HW#A3-1: Show that, given real numbers a and b, $\{x^2 + ax + b \ge 0 \text{ for all } x \in \mathbf{R}\} \Leftrightarrow \{a^2 - b \le 0\}$.

HW#A3-2: Show that $\sqrt{3}$ is irrational.

HW#A3-3: Let $x \in \mathbf{R}$; $f(x) = x$; and $g(x) = e^x$. Show that $f(0) < g(0)$ and $f'(x) < g'(x) \ \forall x > 0$. Now show that $\mathrm{x} \ge 0 \Rightarrow \mathrm{x} < \mathrm{e}^{\mathrm{x}}$.

HW#A3-4: $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ for $n \in \mathbf{N}$. Then, evaluate $\sum_{k=1}^{n} k^2$ and $\sum_{k=1}^{n} k^3$. Use different methods for proof.

HW#A3-5: Show that, given $h \in (0, \infty)$, ,show that $(1+h)^n > (1+nh) \quad \forall n \in \mathbf{N} - \{1\}$. .

HW#A3-6: Show that a total ordering on a set is an equivalence relation. Is the converse true?