ME 550. FOUNDATIONS OF ENGINEERING SYSTEMS ANALYSIS

Appendix #04: Introduction to Measure Theory

Intuitively, Lebesgue measure is the length of an interval or of an at most countable union of intervals on the real line $\Re$. This concept can be extended to $\Re^2$ and $\Re^3$ as areas and volumes, and also to other finite-dimensional spaces. In general, measure is a set function, i.e., an assignment of a number m(A) to each set in a certain class.

Consider the open interval (a,b) whose measure is the length $(b-a)$. Similarly, the length of a countable union of disjoint open intervals can be obtained by summing the lengths of these intervals. Defining sets only in terms of disjoint open intervals is often restrictive. Therefore, we would like to generalize the concept of measure. At this stage, solely for simplicity, we would restrict the treatment of measure to finite intervals and bounded subsets of $\Re$.

- For a finite interval $\mathbf{I}$ (open, closed or semi-open), the measure is equal to the length of the interval, i.e., $m(\mathbf{I}) = \ell(\mathbf{I})$.
- If $\{\mathbf{I_k}\}$ is an at most countable (i.e., finite or countably infinite) sequence of disjoint intervals, then $m\left(\bigcup_{i=1}^{\infty}\mathbf{I}_i\right) = \sum_{i=1}^{\infty}\ell(\mathbf{I}_i)$.
- The measure is translation-invariant. That is, if E is a set for which the measure m is defined and if, for any given $y \in \Re$, $E \oplus y$ is the set $\{x+y : x \in E\}$ obtained by replacing each point x in E by $x+y$, then $m(E \oplus y) = m(E)$.

Outer Measure: For each set $E \subseteq \Re$, consider the countable collection $S = \{\mathbf{I}_k\}$ of open intervals in $\Re$ that cover E, i.e., $E \subseteq \bigcup_{i=1}^{\infty}\mathbf{I}_i$. The outer measure of E is then defined as:

$$\overline{m}(E) = \inf_{all S}\sum_{j=1}^{\infty}\ell(\mathbf{I}_j)$$

Clearly, if $A \subseteq B$, then $\overline{m}(A) \le \overline{m}(B)$. Also, $\overline{m}(\emptyset) = 0$ and each set consisting of a single point has a zero outer measure. The following results are presented from Royden (1989) without proof:

- Result 1: The outer measure of an interval (open or closed or semi-open) is its length.
- Result 2: If $\{A_k\}$ is a sequence of subsets of $\Re$, then $\overline{m}\left(\bigcup_{k=1}^{\infty}A_k\right) \le \sum_{k=1}^{\infty}\overline{m}(A_k)$.
- Result 3: If A is a countable subset (dense or not) of $\Re$, then $\overline{m}(A) = 0$.
- Result 4: If A is an uncountable set, then $\overline{m}(A) \ge 0$. Usually the measure of an uncountable set is greater than 0. However, there are uncountable sets such as the Cantor set whose measure is 0.

Definition A4-18: A set $E \subseteq \Re$ is said to be measurable if, for every $A \subseteq \Re$, the following condition holds: $\overline{m}(A) = \overline{m}(A \cap E) + \overline{m}(A \cap E^c)$. ♦

The fact that $A = (A \cap E) \cup (A \cap E^c)$ implies, by virtue of Result 2, that $\overline{m}(A) \le \overline{m}(A \cap E) + \overline{m}(A \cap E^c)$. Therefore, E is measurable whenever $\overline{m}(A) \ge \overline{m}(A \cap E) + \overline{m}(A \cap E^c)$. Furthermore, because of symmetry, E^c is measurable if and only if E is measurable.

Inner Measure: Assuming that E is a bounded set, we choose an interval $\mathbf{I}^*$ such that $E \subset \mathbf{I}^*$. We define the inner measure $\underline{m}$ as: $\underline{m}(E) = \ell(\mathbf{I}^*) - \overline{m}(\mathbf{I}^* - E)$. The following results are presented without proof:

- Result 5: The inner measure $\underline{m}(E)$ is invariant for every $\mathbf{I}^*$ containing E.
- Result 6: For every bounded set E, then $\underline{m}(E) \le \overline{m}(E) < \infty$.
- Result 7: If E is a finite interval, then $\underline{m}(E) = \overline{m}(E) = \ell(E)$.
- Result 8: A bounded set $E \subseteq \Re$ is measurable if $\underline{m}(E) = \overline{m}(E)$. In that case, we denote the measure as: $m(E) = \underline{m}(E) = \overline{m}(E)$.

Demonstration of the Existence of a Nonmeasurable Set: The usually encountered sets are measurable and, for engineering applications, we may not have to deal with any nonmeasurable sets. Indeed, it is not easy to find a nonmeasurable set. However, from the conceptual point of view, it is important to establish the existence of a non-measurable set. An example of a nonmeasurable set [Royden (1989), pp. 64-65] is given below.

Let $x, y \in [0,1)$. Define the sum Modulo 1 as: $x \oplus y = \begin{cases} x+y & \text{if } (x+y) < 1 \\ x+y-1 & \text{if } (x+y) \ge 1 \end{cases}$

where the operator $\oplus$ can be interpreted as follows: If $\theta = x \oplus y$, then the angle $2\pi\theta$ in radians is the sum modulo addition of two angles $2\pi x$ and $2\pi y$. The sum modulo operator $\oplus$ can also be translated, i.e., $E \oplus y = \{z : z = x \oplus y \text{ for } x \in E\} \quad \forall y \in E$. Furthermore, the operator $\oplus$ is commutative and associative. We establish the following lemma before citing an example of a nonmeasurable set.

Lemma: If $E \subseteq [0,1)$ isa measurable set, then $E \oplus y$ is measurable and m($E \oplus y$) = m(E) $\forall y \in E$.

Proof: Let $E_1 = E \cap [0, 1-y)$ and $E_2 = E \cap [1-y, 1)$ for some $y \in [0,1)$.. Therefore, $E_1 \cap E_2 = \varnothing$ and $E_1 \cup E_2 = E$ which imply that $m(E_1) + m(E_2) = m(E)$. Since the set $E_1 + y$ is measurable and $E_1 \oplus y = E_1 + y$, we argue that $E_1 \oplus y$ is also measurable and m($E_1 \oplus y$) = m($E_1 + y$) = m(E_1). Similarly, $E_2 \oplus y = E_2 + y - 1$ implies that $E_2 \oplus y$ is measurable and m($E_2 \oplus y$) = m(E_2). But $(E_1 \oplus y) \cup (E_2 \oplus y) = E \oplus y$ and $(E_1 \oplus y) \cap (E_2 \oplus y) = \varnothing$. Hence,

$$m((E \oplus y)) = m(E_1 \oplus y) + m(E_2 \oplus y) = m(E_1) + m(E_2) = m(E).$$ ♦

Now we proceed to construct a nonmeasurable set. Let $x, y \in [0,1)$ such that $(x-y)$ is rational. This is an equivalence relation $x \sim y$ because: (i) $x \sim x$ since 0 is rational; (ii) $x \sim y \equiv y \sim x$ since if r is rational, so is $-r$; and (iii) ($x \sim y$ and $y \sim z$) $\Leftrightarrow x \sim z$ because if r and $\tilde{r}$ r' are rational, so is $r + \tilde{r}$. Next, we partition the set $[0,1)$ into equivalence classes such that any two elements of an equivalence class differ by a rational, and any two numbers belonging to different equivalence classes differ by an irrational.

Let us construct the set P that contains exactly one number from each equivalence class and assume that P is a measurable set. Let r_i, $i = 0,1,2,\cdots$, be an enumeration of the rational numbers in $[0,1)$ with $r_0 = 0$. Let us define $P_i = P \oplus r_i \Rightarrow P_0 = P$ and let $x \in P_i \cap P_j$ for $i \ne j$. Therefore, $x = q_i \oplus r_i = q_j \oplus r_j$ with $q_i, q_j \in P$. Then, $q_i - q_j = r_i - r_j$ is a rational and hence $q_i \sim q_j$. Since P has exactly one element from each equivalence class, $q_i = q_j$. That means, for $i \ne j$, $P_i \cap P_j = \varnothing$. Therefore, the collection of sets $\{P_i\}$ is pairwise disjoint. On the other hand, each $x \in [0,1)$ belongs to one and only one of the equivalence classes and therefore must be equivalent to an element of P. But, if x differs from an element of P by a rational r_i, then $x \in P_i$ for some $i \in \{1,2,3,\cdots\}$. Therefore, $\bigcup_{i=1}^{\infty} P_i = [0,1)$. Further, since P_i is a translation modulo 1 of P, we conclude by the lemma, that each P_i is measurable and has the same measure as P. If it is so, $m([0,1)) = \sum_{i=0}^{\infty} m(P_i) = \sum_{i=0}^{\infty} m(P)$ which implies that $m([0,1))$ is either 0 or ∞ depending on whether $m(P)$ is zero or non-zero. But we know that $m([0,1)) = 1$. This is a contradiction. So P is a nonmeasurable set. ♦

Measurable Functions and Convergence almost everywhere

Consider a sequence of functions $\{f_k\}$ defined on a set E. If $\{f_k\}$ converges to a function f at every point on E except possibly on a set $A \subset E$ where $m(A) = 0$, then $\{f_k\}$ is said to converge to f almost everywhere, abbreviated as *a.e.*, on E. The a.e. convergence is conceptually similar to the almost sure (a.s.) convergence of a random sequence.

Definition A4-19: A collection Ψ of subsets of a non-empty set Ω (which may be finite or countably infinite or uncountable) is said to be an algebra in Ω if Ψ satisfies the following properties:

(i) $\Omega \in \Psi$.

(ii) If $E \in \Psi$, then $E^c \in \Psi$ where $E^c \equiv \Omega - E$.

(iii) $E = \bigcup_{i=1}^{n} E_i \in \Psi$ where $E_i \in \Psi$ $\forall$i, then $E \in \Psi$. [This property is as closure under finite union.]

If condition (iii) is relaxed to countable union, i.e., if $E = \bigcup_{i=1}^{\infty} E_i \in \Psi$, then Ψ is called a σ-algebra in Ω.

The duple (Ω, Ψ) is called a *measurable space* where the members of Ψ are called measurable sets in Ω. When there is no confusion, we say Ω is a measurable space instead of (Ω, Ψ). ♦

Remark A4-20: The last two conditions imply that any finite (countable) intersections of events is also an event for an algebra (σ-algebra). ♦

Remark A4-21: In the terminology of probability theory, the non-empty set Ω is called the sample space which is the set of all possible outcomes (of a random experiment) or sample points, and $E \in \Psi$ is called an event. In general, the σ-algebra Ψ is called the event space which is the collection of all possible events. It should be obvious from the above three conditions that any arbitrary subset of Ω may not be qualified as an event. However, the sample space Ω (which is the sure event) and its complement in Ω, namely the empty set $\varnothing$, (which is called the impossible event) are always qualified as events. Every event space must contain these two events. Therefore, for a given sample space, the event space may not be unique. So, the smallest event space which can be obtained as the intersection of all possible event spaces is $\{\varnothing, \Omega\}$. ♦

Remark A4-22: If Ω is a finite set, then there can be only finitely many event spaces, each of which must also be a finite set. In other words, there can be only finitely many different algebras if there are only finitely many elements in Ω. The largest possible event space is the power set 2^{Ω}. However, if the cardinality of Ω is 1, i.e., if there is exactly one experimental outcome, then the only possible event space is $\{\varnothing, \Omega\}$. ♦

Remark A4-23: If Ω is an infinite set, then Ψ can be finite or infinite. This follows from the facts that the smallest Ψ is always finite and the largest Ψ is the power set 2^{Ω} which is infinite if Ω is infinite. Note that, for an infinite Ω, countable or uncountable, it is possible to construct an uncountable Ψ but there does not exist a countably infinite Ψ. ♦

Remark A4-24: It follows from De Morgan's theorem and the last two conditions of σ-algebra that any countable intersection of events is also an event, i.e., if $E = \bigcap_{i=1}^{\infty} E_i \in \Psi$ if $E_i \in \Psi$. ♦

Remark A4-25: In the context of probability theory, each event (i.e., element of the event space) is a measurable set. ♦

Definition A4-20: A nonnegative finitely additive set function μ defined on $\mathfrak{E}$ is called finite iff μ(Ω) is finite. This implies that μ(E) is finite for every E∈$\mathfrak{E}$. Furthermore, μ defined on $\mathfrak{E}$ is called σ-finite iff there exist a sequence $\{\mathrm{E_i}\}$ with $\mathrm{E_i} \in \mathfrak{E}$ such that $\Omega = \bigcup_{\mathrm{i=1}}^{\infty} \mathrm{E_i}$ and $\mu(\mathrm{E_i}) < \infty \ \forall \mathrm{i}$. (Note that Lebesgue measure is σ-finite but not finite.). ♦