ME 550. FOUNDATIONS OF ENGINEERING SYSTEMS ANALYSIS

Appendix #05: Important Theorems for Exchange of Limits, Summation, and Integrals

In many engineering problems, we exchange the orders of limits, infinite sums, and integrals. One should be cautious about these exchanges because such operations may not be always valid and may cause errors under certain circumstances. (Note that there is no problem in exchanging finite summations with limits, infinite summations, and integrals.) Before stating relevant theorems to support these notions, we cite a few examples to demonstrate that such problems do exist in engineering analysis.

Case 1: In general, $\lim_{n\to\infty}\lim_{m\to\infty} f_{m,n} \neq \lim_{m\to\infty}\lim_{n\to\infty} f_{m,n}$

We cite an example. Let $f_{m,n} = \dfrac{m}{m+n}$. Then, for any fixed $n \in \mathbf{N}$, we have

$$\lim_{m\to\infty} f_{m,n} = \lim_{m\to\infty}\frac{m}{m+n} = 1 \;\Rightarrow\; \lim_{n\to\infty}\lim_{m\to\infty} f_{m,n} = 1.$$

However, for any fixed $m \in \mathbf{N}$, we have:

$$\lim_{n\to\infty} f_{m,n} = \lim_{n\to\infty}\frac{m}{m+n} = 0 \;\Rightarrow\; \lim_{n\to\infty}\lim_{m\to\infty} f_{m,n} = 0.$$ ♦

Case 2: In general, $\lim_{n\to\infty} f_n = f$ does not imply $\lim_{n\to\infty}\dot{f}_n = \dot{f}$ where $\dot{f}(t)$ indicates $\frac{d}{dt}\big(f(t)\big)$.

We cite an example. Let $f_n(t) = \dfrac{\sin(nt)}{\sqrt{n}}$ for $t \in \mathbf{R} \equiv (-\infty,\infty)$ and $n \in \mathbf{N}$. Then,

$$f(t) = \lim_{n\to\infty} f_n(t) = \lim_{n\to\infty}\frac{\sin(nt)}{\sqrt{n}} = 0 \;\Rightarrow\; \dot{f}(t) \equiv \frac{d}{dt}\big(f(t)\big) = 0$$

But $\dot{f}_n(t) \equiv \frac{d}{dt}\big(f_n(t)\big) = \frac{d}{dt}\left(\dfrac{\sin(nt)}{\sqrt{n}}\right) = \sqrt{n}\cos(nt) \Rightarrow \lim_{n\to\infty}\dot{f}_n(t) = \lim_{n\to\infty}\sqrt{n}\cos(nt)$ does not exist in $\mathbf{R}$. ♦

Case 3: In general, $\lim\int \neq \int\lim$

We cite an example. Let $f_n(t) = n^2 t\left(1-t^2\right)^n$ for $t \in [0,1]$ and $n \in \mathbf{N}$. Then,

$$f(t) \equiv \lim_{n\to\infty} f_n(t) = \lim_{n\to\infty} n^2 t\left(1-t^2\right)^n = 0 \quad \forall t \in [0,1]$$

$$\text{and } \int_0^1 dt\, f(t) = \int_0^1 dt \lim_{n\to\infty} n^2 t(1-t^2)^n = 0$$

On the other hand, $\int_0^1 dt\, f_n(t) = \int_0^1 dt\, n^2 t\left(1-t^2\right)^n = \frac{n^2}{2(n+1)} \;\Rightarrow\; \lim_{n\to\infty}\int_0^1 dt\, f_n(t) = \lim_{n\to\infty}\frac{n^2}{2(n+1)}$ does not exist in $\mathbf{R}$.

We cite another example where both limits exist but they are unequal. Let $f_n(t) = nt\left(1-t^2\right)^n$ for $t \in [0,1]$, $n \in \mathbf{N}$.

Then, $f(t) \equiv \lim_{n\to\infty} f_n(t) = \lim_{n\to\infty} nt\left(1-t^2\right)^n = 0 \;\; \forall t \in [0,1] \Rightarrow \lim_{n\to\infty}\int_0^1 dt\, f_n(t) = 0$.

On the other hand, $\int_0^1 dt\, f_n(t) = \int_0^1 dt\, nt\left(1-t^2\right)^n = \frac{n}{2(n+1)} \;\Rightarrow\; \lim_{n\to\infty}\int_0^1 dt\, f_n(t) = \lim_{n\to\infty}\frac{n}{2(n+1)} = \frac{1}{2}$.

Both limits exist but they are unequal. ♦

Case 4: In general, $\int_{T_x} dx \int_{T_y} dy\, f(x,y) \neq \int_{T_y} dy \int_{T_x} dx\, f(x,y)$

We cite an example. Let $f(x,y) = \dfrac{x-y}{(x+y)^3}$ for $x \in [0,1]$ and $y \in [0,1]$. Then,

$$\int_0^1 dx \int_0^1 dy \left(\frac{x-y}{(x+y)^3} \right) = \int_0^1 dx \left[\frac{1}{x+y} - \frac{x}{(x+y)^2} \right]_{y=0}^1 = \int_0^1 \frac{dx}{(1+x)^2} = \left. \frac{-1}{(1+x)} \right|_{x=0}^1 = \frac{1}{2} \text{ and}$$

$$\int_0^1 dy \int_0^1 dx \left(\frac{x-y}{(x+y)^3} \right) = \int_0^1 dy \left[\frac{y}{(x+y)^2} - \frac{1}{x+y} \right]_{x=0}^1 = -\int_0^1 \frac{dy}{(1+x)^2} = \left. \frac{1}{(1+y)} \right|_{y=0}^1 = -\frac{1}{2}$$

The reasons for inequality of these two integrals are that (i) the function $f \notin L_1(T_x \times T_y)$, i.e., f is not absolutely integrable on $[0,1] \times [0,1]$; and (ii) f changes sign on its range, i.e., f becomes negative on its range. ♦

We now state several important theorems, without proof, that are important for determining when exchange of limits, infinite summation, and integrals are permissible.

Theorems Stated in Non-Measure-Theoretic Terms

Definition A5-1: A sequence of functions $\{f_k\}$ converges uniformly to a function f on a nonempty set Ω if $\forall \varepsilon > 0\ \exists\, n(\varepsilon) \in \mathbf{N}$ such that, for all $k \geq n$, $|f_k(t) - f(t)| < \varepsilon \quad \forall t \in \Omega$.

Theorem A5-1: Let $\alpha(\bullet)$ be a monotonically increasing function on $\mathbf{T} \subseteq \mathbf{R}$, i.e., $\alpha(x) \geq \alpha(y)\ \forall x > y$. Let $\{\varphi_k\}$ be a sequence of Riemann-Stieltjes integrable functions on $\mathbf{T}$ w.r.t. α, i.e., $\int_T d\alpha(t) |\varphi_k(t)| < \infty$. If $\varphi_k \to \varphi$ uniformly on $\mathbf{T}$, then

- φ is Riemann-Stieltjes integrable on $\mathbf{T}$ w.r.t. α, i.e., $\int_T d\alpha(t) |\varphi(t)| < \infty$.
- $\lim\limits_{k\to\infty} \int_T d\alpha(t)\varphi_k(t) = \int_T d\alpha(t)\varphi(t)$ (i.e., $\lim\int = \int\lim$) ♦

Corollary to Theorem A5-1: If the series $s_k(t) \equiv \sum\limits_{n=1}^{k} \varphi_n(t)$ converges uniformly to $s(t)$ on $\mathbf{T}$, then

- $\lim\limits_{k\to\infty} \sum\limits_{n=1}^{k} \int_{\mathbf{T}} d\alpha(t)\varphi_n(t) = \int_{\mathbf{T}} d\alpha(t) s(t)$ (i.e., $\sum\limits_1^\infty \int = \int \sum\limits_1^\infty$) ♦

Theorem A5-2: Let $\varphi_k \to \varphi$ uniformly on $\mathbf{T} \subseteq \mathbf{R}$, and let τ be a limit point on $\mathbf{T} \subseteq \mathbf{R}$. If $\lim\limits_{t\to\tau} \varphi_k(t) = \nu_k\ \forall k \in N$, then the sequence $\{\nu_k\}$ converges and $\lim\limits_{t\to\tau} \varphi(t) = \lim\limits_{t\to\tau} \nu_k$. In other words, $\lim\limits_{t\to\tau} \lim\limits_{k\to\infty} \varphi_k(t) = \lim\limits_{k\to\infty} \lim\limits_{t\to\tau} \varphi_k(t)$. ♦

Theorem A5-3: Let $\{\varphi_k\}$ be a sequence of functions that are differentiable on $\mathbf{T} \subseteq \mathbf{R}$, and let $\{\varphi_k(\tau)\}$ converge to $\varphi(\tau)$ for some $\tau \in \mathbf{T}$. If the derivative $\dot{\varphi}_k(\tau) \equiv \left. \frac{d\varphi_k(t)}{dt} \right|_{t=\tau}$ converges uniformly on $\mathbf{T}$, then

- $\{\varphi_k\}$ converges to φ uniformly on $\mathbf{T}$.
- $\lim\limits_{k\to\infty} \dot{\varphi}_k(t) = \dot{\varphi}(t)$ on $\mathbf{T}$ (i.e., $\lim \frac{d}{dt} = \frac{d}{dt} \lim$) ♦

Theorem A5-4 (Fubini's Theorem): Let $\varphi : \mathbf{R}^{\ell} \times \mathbf{R}^{m} \to \mathbf{R}$.. Let $\varphi^{y}(x) = \varphi(x,y)$ for any fixed $y \in \mathbf{R}^{m}$, and $\varphi_{x}(y) = \varphi(x,y)$ for any fixed $x \in \mathbf{R}^{\ell}$. Then, the following conditions (i) and (ii) hold:

(i) If $\left(\int\limits_{\mathbf{R}^{m}} d\mu(\mathrm{x}) \int\limits_{\mathbf{R}^{\ell}} \mathrm{d}\nu(\mathrm{y}) \left|\varphi_{x}(y)\right| < \infty \right)$ or if $\left(\int\limits_{\mathbf{R}^{m}} dy \int\limits_{\mathbf{R}^{\ell}} d\mathrm{x} \left|\varphi(x,y)\right| < \infty \right)$, then

(a) $\varphi_{x} \in L_{1}(\mathbf{R}^{m})$ and $\zeta(x) \equiv \int\limits_{\mathbf{R}^{m}} dy\, \varphi_{x}(y) \in L_{1}(\mathbf{R}^{\ell})$.

(b) $\varphi^{y} \in L_{1}(\mathbf{R}^{\ell})$ and $\psi(y) \equiv \int\limits_{\mathbf{R}^{\ell}} dx\, \varphi^{y}(x) \in L_{1}(\mathbf{R}^{m})$.

(c) $\int\limits_{\mathbf{R}^{\ell}} d\mathrm{x} \int\limits_{\mathbf{R}^{m}} \mathrm{dy} \left|\varphi(x,y)\right| = \int\limits_{\mathbf{R}^{m}} dy \int\limits_{\mathbf{R}^{\ell}} d\mathrm{x} \left|\varphi(x,y)\right| = \int\limits_{\mathbf{R}^{\ell} \times \mathbf{R}^{m}} dx\, dy \left|\varphi(x,y)\right|$.

(ii) If $\varphi(x,y) \geq 0$ almost everywhere on $X \times Y$, then

$\int\limits_{\mathbf{R}^{\ell}} d\mathrm{x} \int\limits_{\mathbf{R}^{m}} \mathrm{dy} \left|\varphi(x,y)\right| = \int\limits_{\mathbf{R}^{m}} dy \int\limits_{\mathbf{R}^{\ell}} d\mathrm{x} \left|\varphi(x,y)\right| = \int\limits_{\mathbf{R}^{\ell} \times \mathbf{R}^{m}} dx\, dy \left|\varphi(x,y)\right|$. ♦

Theorems Stated in Measure-Theoretic Terms

Theorem A5-5 (Lebesgue-Monotone Convergence Theorem): Let $\{\varphi_{k} : X \to [0,\infty]\}$ be a sequence of monotonically increasing Lebesgue-measurable functions on a measure space (X, Σ, μ) such that $\lim\limits_{k\to\infty} \varphi_{k} = \varphi$ almost everywhere on X. Then,

- φ is a Lebesgue-measurable function.
- $\lim\limits_{k\to\infty} \int\limits_{X} d\mu\, \varphi_{k} = \int\limits_{X} d\mu\, \varphi$ (i.e., $\lim \int = \int \lim$) ♦

Corollary to Theorem A5-5: Let $\{\varphi_{k} : X \to [0,\infty]\}$ almost everywhere on X be a sequence of Lebesgue-measurable functions on a measure space (X, Σ, μ), and let $s_{k} \equiv \sum\limits_{n=1}^{k} \varphi_{n}$ such that $\lim\limits_{k\to\infty} s_{k} = s$ almost everywhere on X. Then,

- $\lim\limits_{k\to\infty} \sum\limits_{n=1}^{k} \int\limits_{X} d\mu\, \varphi_{n} = \int\limits_{X} d\mu\, s$ (i.e., $\sum\limits_{1}^{\infty} \int = \int \sum\limits_{1}^{\infty}$) ♦

Theorem A5-6 (Fatou's Lemma): Let $\{\varphi_{k} : X \to [0,\infty]\}$ be a sequence of Lebesgue-measurable functions on a measure space (X, Σ, μ) such that $\lim\limits_{k\to\infty} \varphi_{k} = \varphi$ almost everywhere on X. Then,

- $\int\limits_{X} d\mu \liminf \varphi_{k} \leq \liminf \int\limits_{X} d\mu\, \varphi_{k}$ ♦

Theorem A5-7 : Let $\varphi : X \to [0,\infty]$ almost everywhere on X be a Lebesgue-measurable function on a measure space (X, Σ, μ), and let $\psi(E) \equiv \int\limits_{E} d\mu\, \varphi \quad \forall E \in \Sigma$. Then,

- ψ is a measure on Σ.
- $\int\limits_{X} d\psi\, f = \int\limits_{X} d\mu\, f\varphi$ for every Lebesgue-measurable function f on a measure space (X, Σ, μ). ♦

Theorem A5-8 : Let φ be a Lebesgue-measurable function on a measure space (X,Σ,μ). If $\varphi \in L_1(\mu)$, i.e., φ is absolute-integrable w.r.t. the measure μ. Then, $\left|\int_E d\mu\, \varphi\right| \le \int_E d\mu\, |\varphi| \quad \forall E \in \Sigma.$ ♦

Theorem A5-9 (Lebesgue Dominated Convergence Theorem): Let $\{\varphi_k\}$ be a sequence of Lebesgue-measurable functions on a measure space (X,Σ,μ) such that $\lim_{k\to\infty} \varphi_k = \varphi$ almost everywhere on X. If $\exists\ \psi \in L_1(\mu)$ such that $|\varphi_k| \le \psi \quad \forall k \in \mathbf{N}$ almost everywhere on X, then

- $\varphi \in L_1(\mu)$
- $\lim_{k\to\infty} \int_X d\mu |\varphi_k - \varphi| = 0$
- $\lim_{k\to\infty} \int_X d\mu\, \varphi_k = \int_X d\mu\, \varphi$ (i.e., $\lim\int \ = \int\lim$) ♦

Theorem A5-10 (Fubini's Theorem): Let (X,Σ,μ) and (Y,T,ν) be $\sigma-$finite measure spaces. Let φ be a ($\Sigma\times\mathrm{T}$)-measurable function on $X\times Y$ with the product measure $\pi \equiv \mu\times\nu$. Let $\varphi^y(x) \equiv \varphi(x,y)$ for any fixed $y\in Y$, and $\varphi_x(y) \equiv \varphi(x,y)$ for any fixed $x\in X$. Then, the following conditions (i) and (ii) hold:

(i) If $\left(\int_X d\mu(\mathrm{x})\int_Y \mathrm{d}\nu(\mathrm{y})|\varphi_x(y)| < \infty\right)$ or if $\left(\int_Y d\nu(y)\int_X d\mu(\mathrm{x})|\varphi(x,y)| < \infty\right)$, then

(a) $\varphi_x \in L_1(\nu)$ and $\zeta(x) \equiv \int_Y d\nu(y)\varphi_x(y) \in L_1(\mu)$.

(b) $\varphi^y \in L_1(\mu)$ and $\psi(y) \equiv \int_Y d\mu(x)\varphi^y(x) \in L_1(\nu)$.

(c) $\int_X d\mu(\mathrm{x})\int_Y \mathrm{d}\nu(\mathrm{y})|\varphi(x,y)| = \int_Y d\nu(y)\int_X d\mu(\mathrm{x})|\varphi(x,y)| = \int_{X\times Y} d\pi(x,y)\,|\varphi(x,y)|$.

(ii) If $\varphi(x,y) \ge 0$ almost everywhere on $X\times Y$, then

$\int_X d\mu(\mathrm{x})\int_Y \mathrm{d}\nu(\mathrm{y})|\varphi(x,y)| = \int_Y d\nu(y)\int_X d\mu(\mathrm{x})|\varphi(x,y)| = \int_{X\times Y} d\pi(x,y)\,|\varphi(x,y)|$. ♦