# ME 550. FOUNDATIONS OF ENGINEERING SYSTEMS ANALYSIS

## Chapter 1: Topological Structure

Topology is a vast subject and it has several different branches such as Point-set Topology, Algebraic Topology, Differential Topology, and Combinatorial Topology. Rudimentary concepts of point-set topology are presented in this section. The presented notion of topology allows generalization of open sets and continuity of functions beyond metric spaces described in the next subsection.

### *1.1 Metric Spaces*

**Definition 1.1-1**: Let $S$ be a nonempty set and the pair $\langle S,d\rangle$ is called a metric space where $d$ is a metric (or a distance function) $d: S\times S\to[0,\infty)$ defined, for every $x,y,z\in S$ as:

- $d(x,y)=0$ if and only if $x=y$ — Strict positivity
- $d(x,y)=d(y,x)$ — Symmetry
- $d(x,z)\le d(x,y)+d(y,z)$ — Triangular inequality ♦

**Exercise#1.1-1**: From the triangular inequality property, show that $|d(x,z)-d(y,z)|\le d(x,y)$. ♦

**Example 1.1-1**: For the real line **R** and the complex plane **C**, a metric is defined as: $d(x,y)=|x-y|$. For the Euclidean space $\mathbf{R}^n$ and the unitary space $\mathbf{C}^n$, a metric is defined as: $d(x,y)=\sqrt{\sum_{k=1}^{n}|x_k-y_k|^2}$. ♦

**Example 1.1-2**: Let $S$ be the set of all bounded sequences of complex numbers. Let $x=\{x_1,x_2,x_3,\cdots\}$ be a sequence of complex numbers such that, for all $j=1,2,3,\cdots$, we have $|x_j|\le C_x\in[0,\infty)$ where $C_x$ may depend on $x$ but not on $j$. If the metric is chosen as: $d(x,y)=\sup_{j\in\mathbf{N}}|x_j-y_j|$, this space is called the $\ell_\infty$-space. ♦

**Exercise#1.1-2**: Show that the metric in Example 1.1-2 satisfies the triangular inequality property. ♦

**Example 1.1-3**: Let $S=\mathrm{C}[a,b]$ be the set of all real-valued continuous functions defined on the closed interval $[a,b]\subset\mathbf{R}$. A metric on $\mathrm{C}[a,b]$ could be chosen as: $d(x,y)=\underset{t\in[a,b]}{Max}|x(t)-y(t)|$. ♦

**Example 1.1.4**: The distance function $d(x,y)=\int_a^b dt\,|x(t)-y(t)|^p$ is valid on the space $L_p[a,b]$ of $p^{th}$-power integrable functions on the interval [a,b], where $p\in[1,\infty)$. ♦

**Example 1.1-5**: On the space of $n\times m$ real matrices, a metric could be chosen as: $d(A,B)=Trace\big((A-B)^T(A-B)\big)$. ♦

**Example 1.1-6**: Let $d(x,y)$ be a metric on a space S. Then, $\tilde{d}(x,y)=\dfrac{d(x,y)}{1+d(x,y)}$ is also a metric on S. ♦

**Example 1.1-7**: Let $d(x,y)$ be a metric on $S$. Then, $\breve{d}(x,y)=\min(1,\ d(x,y))$ is also a metric on S. ♦

**Example 1.1-8**: Consider any nonempty set S and define the discrete metric on S as: $d(x,y)=\begin{cases}0 & \text{if } x=y\\ 1 & \text{if } x\ne y\end{cases}$.
The space $\langle S,d\rangle$ is called the discrete metric space. ♦

**Definition 1.1-2**: Let $\langle S,d\rangle$ be a metric space and let a nonempty set $A\subseteq S$. The diameter of $A$ is defined as: $diam(A)=\sup\{d(x,y):x,y\in A\}$. The distance of a point $x\in S$ from $A$ is defined as: $d(x,A)=\inf\{d(x,y):y\in A\}$. ♦

Let us consider a metric space $\langle S, d \rangle$. Let $E \subseteq S$ be nonempty. We introduce the following concepts on the metric space $\langle S, d \rangle$:

- A neighborhood of a point $x \in E$ is a nonempty set $B_r(x) \equiv \{y \in S : d(x, y) < r \in (0, \infty)\}$. Often $B_r(x)$ is called the open ball of radius $r$ with center at $x$.
- A point $x \in S$ is a limit point of $E$ if every neighborhood of $x$ contains a point $y \neq x$ such that $y \in E$.
- If $x \in E$ and $x$ is not a limit point of $E$, then $x$ is an isolated point of $E$.
- A set $E$ is closed in $S$ if every limit point of $E$ is contained in $E$.
- A point $x$ is an interior point of $E$ if there exists a neighborhood $B_r(x) \subseteq E$.
- The complement of $E$ in $S$, denoted as $E^c$ and also as $S - E$, is the set all points $y$ in $S$ such that $y \notin E$.
- A set $E$ is open in $S$ if every point of $E$ is an interior point of $E$.
- A set $E$ is perfect in $S$ if $E$ is closed in $S$ and every point of $E$ is a limit point of $E$.
- A set $E$ is bounded if there exists a real number $M \in (0, \infty)$ and a point $y \in S$ such that $d(x, y) < M \quad \forall x \in E$.
- A set $E$ is dense in $S$ if every point of $S$ is a limit point of $E$ or a point of $E$ (or both). ♦

**<u>Example 1.1-9</u>**: Consider the following subsets of the two-dimensional real plane $\mathbf{R} \times \mathbf{R}$:

| Description | closed | open | perfect | bounded |
|---|---|---|---|---|
| $\left\{(x, y) \in \mathbf{R} \times \mathbf{R} : \sqrt{x^2 + y^2} < 1\right\}$ | no | yes | no | yes |
| $\left\{(x, y) \in \mathbf{R} \times \mathbf{R} : \sqrt{x^2 + y^2} \leq 1\right\}$ | yes | no | yes | yes |
| Any finite subset of $\mathbf{R} \times \mathbf{R}$ | yes | no | no | yes |
| The set $J \times J$ | yes | no | no | no |
| $\left\{\frac{1}{k} : k = 1,2,3,\cdots\right\} \times \left\{\frac{1}{k} : k = 1,2,3,\cdots\right\}$ | no | no | no | yes |
| The entire set $\mathbf{R} \times \mathbf{R}$ | yes | yes | yes | no |
| The segment $(a,b) \times (a,b)$ | no | yes | no | yes |
| The segment $[a,b] \times [a,b]$ | yes | no | yes | yes |
| The segment $[a,b) \times (a,b]$ | no | no | no | yes |
| The set $\mathbf{Q} \times \mathbf{Q}$ | no | no | no | no |
| The empty set $\varnothing$ | yes | yes | yes | yes |

♦

**<u>Example 1.1-10</u>**: The set $\mathbf{Q}$ of rational numbers is dense in the set $\mathbf{R}$ of real numbers. ♦

**<u>Definition 1.1-3</u>**: Let $\{x_k\}$ be a sequence in a metric space $\langle S, d \rangle$. Then, $\{x_k\}$ converges in S if there exists a point x $\in$S, called the limit point of the sequence, with the following property: $\forall \varepsilon > 0$ there exists a positive integer $K$ such that $\forall k \geq K \quad d(x_k, x) < \varepsilon$. Equivalently, $d(x_k, x) \to 0$ as $k \to \infty$. ♦

Note that if $x \notin S$, we cannot say that the sequence $\{x_k\}$ converges in $S$ even though $d(x_k, x) \to 0$ as $k \to \infty$. In that case, we relax Definition 1-1-3 as follows:

**Definition 1.1-4**: A Cauchy sequence $\{x_k\}$ in a metric space $\langle S,d\rangle$ is defined as follows: $\forall\varepsilon>0$ there exists a positive integer $M$ such that $d(x_k,x_\ell)<\varepsilon \quad \forall k,\ell\geq M$. ♦

**Remark 1.1.1**: A convergent sequence is a Cauchy sequence but the converse is not true, in general. ♦

**Example 1.1-11**: Let $S=(0,1]$. Then, the sequence $\left\{\frac{1}{k}:k=1,2,3,\cdots\right\}$ does not converge in $S$ but it is a Cauchy sequence in $S$. ♦

**Definition 1.1-5**: Let $\langle X,d_x\rangle$ and $\langle Y,d_Y\rangle$ be two metric spaces. A function $f:X\to Y$ is said to be continuous (more precisely, $d_x-d_y$ continuous) at a point $x\in X$ if $\forall\varepsilon>0\ \ \exists\delta(\varepsilon,x)>0$ such that $d_x(x,z)<\delta$ $\Rightarrow d_y(f(x),f(z))<\varepsilon\ \ \forall z\in X$. If $f$ is continuous at every $x\in X$, then $f(\cdot)$ is continuous in $X$. ♦

**Definition 1.1-6**: Let $\langle X,d_x\rangle$ and $\langle Y,d_Y\rangle$ be two metric spaces. A function $f:X\to Y$ is said to be uniformly continuous in X if $\forall\varepsilon>0\ \ \exists\delta(\varepsilon)>0$ such that $d_x(x,z)<\delta \ \Rightarrow d_y(f(x),f(z))<\varepsilon\ \ \forall x,z\in X$. ♦

**Definition 1.1-7**: A metric space is called complete if every Cauchy sequence converges in the metric space.

**Definition 1.1-8**: A sequence $\{x_k\}$ of real numbers is said to be:

- monotonically increasing if $x_k\leq x_{k+1}$; and strictly monotonically increasing if $x_k<x_{k+1}$.
- monotonically decreasing if $x_k\geq x_{k+1}$; and strictly monotonically decreasing if $x_k>x_{k+1}$. ♦

**Definition 1.1-9**: The limit superior of a sequence $\{x_n\}$ is defined as: $\limsup x_n=\inf_n \sup_{k\geq n} x_n$ and the limit inferior of a sequence $\{x_n\}$ is defined as: $\liminf x_n=\sup_n \inf_{k\geq n} x_n$. A sequence is said to have a limit if its limit superior is equal to the limit inferior. for example, the sequence $\{(-1)^n\}$ does not have a limit. Why? ♦

**Exercise#1.1-3**: Show that uniform continuity implies continuity but the converse may not be true. ♦

**Exercise#1.1-4**: Let $X=(0,\infty)$ and $Y=(0,\infty)$. Let $d_x(p,q)=|p-q|$ and $d_y(p,q)=|p-q|$. Let the function $f:X\to Y$ be defined as: $f(t)=\begin{cases}\frac{1}{t} \text{ for } t\in(0,1)\\ 1 \text{ for } t\in[1,\infty)\end{cases}$. Verify that $f(\cdot)$ is continuous but not uniformly continuous. ♦

**Definition 1.1-10**: A function $f$: [a, b] → **R**, where **R** = (-∞, ∞), is said to be absolutely continuous on [a, b] if $\forall\varepsilon>0\ \ \exists\delta(\varepsilon)>0$ such that $\sum_{i=1}^{n}|f(\tilde{x}_i)-f(x_i)|<\varepsilon$ for every finite collection of pairwise disjoint intervals $\{(x_i,\tilde{x}_i):i=1,2,\cdots,n\}$ in [a, b] under the constraint $\sum_{i=1}^{n}|\tilde{x}_i-x_i|<\delta$. ♦

**Definition 1.1-11**: Let $f$: [a, b] → **R**, where **R** = (-∞, ∞) and let $\mathcal{P}$ be the set of all finite collections $\{(x_i,\tilde{x}_i):i=1,\cdots,n\}$ of disjoint intervals in [a, b]. Then, the (total) variation $V_a^b(f)$ of a function $f$ on [a, b] is defined as: $V_a^b(f)=\sup\left\{\sum_{i=1}^{n}|f(\tilde{x}_i)-f(x_i)|\right\}$ over all partitions P ∈ $\mathcal{P}$. If $V_a^b(f)$ is finite, then $f$ is of bounded variation on [a, b]. The set of all functions of bounded variation on [a, b] is denoted as $BV[a,b]$. ♦

**Remark 1.1-2**: If $f$ is absolutely continuous on [a, b], then $f\in BV[a,b]$. ♦

## *1.2 Point-set Topology*

Only the rudimentary concepts of point-set topology are presented in this section This notion allows generalization of open sets and continuity of functions beyond metric spaces described in Section 1.1. .

**Definition 1.2-1**: Let $X$ be a nonempty set and let $\Im$ be a collection of subsets of $X$ such that:

- $\varnothing \in \Im$ and $X \in \Im$.
- If $S_k \in \Im$ for $k = 1,2,\cdots,n$, then $\bigcap_{k=1}^{n} S_k \in \Im$ finite intersection
- If $S_\alpha \in \Im$ for $\alpha \in I$ where $I$ is the index set, then $\bigcup_{\alpha \in I} S_\alpha \in \Im$ arbitrary union

Then, $\Im$ is a topology of $X$ and $\langle X, \Im \rangle$ is a topological space. Each member of $\Im$ is called a $\Im$-open set in $X$.

♦

**Definition 1.2-2**: Let $\langle X, \Im \rangle$ be a topological space. Then the complement of every $\Im$-open set in $X$ is said to be $\Im$**-closed** in $X$. That is, if $S \in \Im$, then $S^c \equiv X - S$ is $\Im$-closed in $X$. In other words, $S$ is $\Im$-open in $X$ if and only if $S^c \equiv X - S$ is $\Im$-closed in $X$. ♦

In view of Definition 1.2.2, an alternative form of of Definition 1.2.1 is given below.

**Definition 1.2-1a**: Let $X$ be a nonempty set and let $\Im$ be a collection of subsets of $X$ such that:

- $\varnothing \in \Im$ and $X \in \Im$.
- If $S_k \in \Im$ for $k = 1,2,\cdots,n$, then $\bigcup_{k \in \{1,2,\cdots,n\}} S_k \in \Im$ finite union
- If $S_\alpha \in \Im$ for $\alpha \in I$ where $I$ is the index set, then $\bigcap_{\alpha \in I} S_k \in \Im$ arbitrary intersection

Then, $\Im$ is a topology of $X$ and $\langle X, \Im \rangle$ is a topological space. Each member of $\Im$ is called a $\Im$-closed set in $X$.

♦

**Definition 1.2-3**: The standard or **usual topology** $\langle \Re, \mathsf{U} \rangle$ is defined with $X = \Re \equiv (-\infty, \infty)$ and $\mathsf{U}$ is the smallest topology (i.e., intersection of all topologies) that contains all open intervals in $\Re$. A set $G \subseteq \Re$ is said to be $\mathsf{U}$-open (i.e., open relative to the usual topology $\mathsf{U}$) if either $G = \varnothing$ or, for $G \neq \varnothing$, $\forall p \in G$ there exists an open interval $(a,b) \subset G$ such that $p \in (a,b)$. ♦

**Definition 1.2-4**: Let $\langle X, \Im \rangle$ be a topological space and let $p \in X$. Then, $B \subseteq X$ is called a $\Im$**-neighborhood** of $p \in X$ if there exists a $\Im$**-open** set $G$ such that $p \in G \subseteq B$. ♦

**Remark 1.2-1**: Note that, in the topological sense, a $\Im$-neighborhood of a point $p \in X$ need not be a $\Im$-open set in $X$. However, a $\Im$-open set is a $\Im$-neighborhood of each of its points. ♦

**Definition 1.2-5**: A basis on a topological space $\langle X, \Im \rangle$ is a collection $\mathcal{B}$ of $\Im$-open sets, called basis elements, such that (1) For each element $p \in X$, there exists a basis element $B \in \mathcal{B}$ such that $p \in B$.

(2) If $p \in B_1 \cap B_2$, where $B_1, B_2 \in \mathcal{B}$, then there exists a $B \in \mathcal{B}$ such that $p \in B \subseteq B_1 \cap B_2$.

If $\mathcal{B}$ satisfies the above two conditions, then we define the topology $\Im$ generated by $\mathcal{B}$ as follows.

For a $\Im$-open set $Y \subseteq X$ and for each $p \in Y$, there exists a basis element $B \in \mathcal{B}$ such that

$p \in B$ and $B \subseteq Y$. Note that each basis element is itself a $\Im$-open set. ♦

**Remark 1.2-2**: Every $\Im$-open set $Y \subseteq X$ in a topological space $\langle X, \Im \rangle$ can be expressed as a union of basis elements. This expression for $Y \subseteq X$ is not unique. Thus, the usage of the term basis in topology differs from that in linear algebra, the equation expressing a given vector as a linear combination of basis vectors is unique. The following lemma addresses this point. ♦

**Lemma 1.2.1**: Let $\langle X, \Im \rangle$ be a topological space and let $\mathcal{B}$ be a basis for $\langle X, \Im \rangle$. Then, $\Im$ equals the collection of all unions of elements of $\mathcal{B}$.

**Proof**: Given a collection of elements of the basis $\mathcal{B}$, they are also elements of $\Im$. Since $\Im$ is a topology, their union is also in $\Im$. So, the collection of all unions of elements of $\mathcal{B}$ is a subset of $\Im$

Conversely, given $Y \in \Im$, let us choose a basis element $\mathrm{B}_\mathrm{p} \in \mathcal{B}$ for each $p \in Y$ such that $\mathrm{p} \in \mathrm{B}_\mathrm{p} \subseteq Y$. Then, $Y \subseteq \bigcup_{p \in Y} \mathrm{B}_\mathrm{p}$ and hence each $Y$ in $\Im$ is a subset of a union of elements of $\mathcal{B}$. Therefore, $\Im$ is a subset of the collection of all unions of elements of $\mathcal{B}$. ♦

**Definition 1.2-6**: Let $S \subset X$ where $\langle \mathrm{X}, \Im \rangle$ is a topological space. Then, a point $p \in X$ is called a **cluster point** of $S$ if every $\Im$-neighborhood of $\mathrm{p}$ contains at least one point of $S$ other than $p$. In other words, $\mathrm{p}$ is a cluster point of $\mathrm{S}$ if and only if the following condition holds: $\mathrm{B}$ is a $\Im$-neighborhood of $\mathrm{p}$ implying that $(B - \{p\}) \cap S \neq \varnothing$. ♦

<u>**Example 1.2-1**</u>: Consider the open interval $(0,1) \subset \Re$. In the usual topology $\langle \Re, \mathrm{U} \rangle$, both 0 and 1 are cluster points of (0,1). Furthermore, every point of (0,1) is a cluster point of (0,1). ♦

**Definition 1.2-7**: Let $\langle X, \Im \rangle$ and $\langle Y, \vartheta \rangle$ be two topological spaces. Then a mapping $f : X \to Y$ is said to **continuous** (more precisely, $\Im - \vartheta$ continuous) if the inverse image $f^{-1}(\Phi)$ is $\Im$-open in $X$ for every $\vartheta -$open set $\Phi$ in $Y$. ♦

**Definition 1.2-8**: A $\Im - \vartheta$ continuous mapping $f : X \to Y$ is $\Im - \vartheta$**-bicontinuous** if $f[\Theta]$ is a $\vartheta -$open set in $Y$ for every $\Im$-open set $\Theta$ in $X$. A bijective and $\Im - \vartheta$-bicontinuous mapping $f : X \to Y$ is called a $\Im - \vartheta$-**homeomorphism** of $X$ and $Y$. ♦

**Remark 1.2-2**: Two topological spaces are equivalent if they are homeomorphic. ♦

**Definition 1.2-9**: A topological space $\langle X, \Im \rangle$ is called a **Hausdorff** ($T_2$) space if, for every pair of distinct points $x$ and $y$, i.e., $x, y \in X$ and $x \neq y$, there exists $\Im$-neighborhoods $B_x$ and $B_y$ such that $B_x \cap B_y = \varnothing$. ♦

**Example 1.2-2**: The usual topology $\langle \Re, \mathrm{U} \rangle$ where the collection of all open subsets (defined in the usual sense) of $\Re = (-\infty, \infty)$ is Hausdorff. ♦

**Example 1.2-3**: Let $X = \{a,b,c\} \subset \Re$ and $\Im = \{\varnothing, \{a,b\}, \{c\}, X\}$. Clearly, the topological space is not Hausdorff because $a$ and $b$ are distinct points of $X$ that do not have disjoint $\Im$-neighborhoods.

**Definition 1.2-10**: Let $\langle X, \Im \rangle$ be a topological space and $Y \subseteq X$. The $\Im$**-relative topology** of $Y$, denoted as $\Im_Y$, is defined as: $\Im_Y = \{G \cap Y : G \in \Im\}$. Then, $\langle Y, \Im_Y \rangle$ is called a subspace of $\langle X, \Im \rangle$. ♦

**Exxercise 1.2-1**: Show that $\Im_Y$ is a topology of $Y$. ♦

**Example 1.2-4**: Let $X = \{a,b,c\} \subset \Re$ and $\Im = \{\varnothing, \{a,b\}, \{c\}, X\}$. Clearly, the topological space $\langle X, \Im \rangle$ is not Hausdorff because $a$ and $b$ are distinct points of $X$ that do not have disjoint $\Im$-neighborhoods. ♦

**Example 1.2-5**: Let $Y=(0,1)\subset\Re$. Consider the relative topology $\langle Y,U_Y\rangle$ in which the interval $(\frac{1}{2},1)$ is open in $\langle Y,U_Y\rangle$. Although $[\frac{1}{2},1)$ is not closed in $\langle\Re,\mathsf{U}\rangle$ but $[\frac{1}{2},1)$ is closed in $\langle Y,U_Y\rangle$ because $[\frac{1}{2},1)$ is the complement of the $U_Y$-open set $(0,\frac{1}{2})$ in $Y$. The set $\{\frac{1}{k}:k\in\mathbf{N}\}$ is closed in the relative topology $\langle Y,U_Y\rangle$, $\langle\Re,\mathsf{U}\rangle$ because it has no cluster points in $Y$. However, $\{\frac{1}{k}:k\in\mathbf{N}\}$ is not closed in the usual topology $\langle\Re,\mathsf{U}\rangle$ because the cluster point 0 is not contained in $\{\frac{1}{k}:k\in\mathbf{N}\}$ ♦

Next, we present three important results:

**Result 1.2-1**: The topological spaces $\langle\Re,\mathsf{U}\rangle$ and $\langle(0,1),\mathsf{U}_{(0,1)}\rangle$ are homeomorphic. This result follows by constructing a bijective and bicontinuous function $f:(0,1)\to\Re$ such as $f(x)=\frac{2x-1}{x(x-1)}$. ♦

**Result 1.2-2**: If $I_1$ and $I_2$ are two $\mathsf{U}$-open intervals in $\Re$, the spaces $\langle \mathrm{I}_1,\mathsf{U}_{\mathrm{I}1}\rangle$ and $\langle I_1,\mathsf{U}_{I2}\rangle$ are homeomorphic. ♦

**Result 1.2-3**: If $I_1$ and $I_2$ are two $\mathsf{U}$-closed intervals in $\Re$, the spaces $\langle I_1,\mathsf{U}_{I1}\rangle$ and $\langle I_1,\mathsf{U}_{I2}\rangle$ are homeomorphic. ♦

## Compactness in a Topological Space

**Definition 1.2-11**: A metric space $\langle S,d\rangle$ is sequentially compact if every sequence in $S$ has a convergent subsequence. ♦

**Example 1.2-6**: The sequence $\{1,\frac{1}{2},3,\frac{1}{4},5,\frac{1}{6},\cdots\}$ has a convergent subsequence $\left\{\frac{1}{2^k}:k\in\mathbf{N}\right\}$ in $\Re$. Note that the sequence $\{1,\frac{1}{2},3,\frac{1}{4},5,\frac{1}{6},\cdots\}$ itself is not convergent in $\Re$ and that it contains many subsequences like $\{1,3,5,7,\cdots\}$ which are not convergent. ♦

**Example 1.2-7**: The set $(0,1]$ is not compact in $\langle\Re,\mathsf{U}\rangle$ because the sequence $\{\frac{1}{k}:k\in\mathbf{N}\}$ does not have a subsequence with a limit point in $(0,1]$. ♦

**Definition 1.2-12**: Let $\langle S,\Im\rangle$ be a topological space and let $E\subseteq S$. Let $\Sigma=\{S_\alpha:\alpha\in I\}$ be a collection of subsets of $S$ where $I$ is an index set (which nonempty, and finite or countable or uncountable). Then, $\Sigma$ is said to be a covering of $E$ if $E\subseteq\bigcup_{\alpha\in I}S_\alpha$. If $\Sigma_1$ is a covering of $E$ and $\Sigma_2$ is a covering of $E$ such that $\Sigma_2\subseteq\Sigma_1$, then $\Sigma_2$ is a subcovering of $\Sigma_1$. ♦

**HW 1.2-2**: Let $\Sigma_1=\{(0,\frac{k}{k+1}):k\in\mathbf{N}\}$. Verify that $\Sigma_1$ is a $\mathsf{U}$-open covering of $(0,1)$ ♦

**Example 1.2-8**: If $\Sigma_2=\left\{\left(0,\frac{4k+3}{4(k+1)}\right):k\in\mathbf{N}\right\}$ is a subcovering of $\Sigma_1$, then $\Sigma_1=\{(0,\frac{k}{k+1}):k\in\mathbf{N}\}$ ♦

**Definition 1.2-13**: Let $\langle S,\Im\rangle$ be a topological space. A covering $\Sigma$ of $E\subseteq S$ is said to be a $\Im$-open covering of E if every member of $\Sigma$ is a $\Im$-open set. A covering $\Sigma$ of a set $E$ is finite if card($\Sigma$) is finite. ♦

**Definition 1.2-14**: A topological space $\langle S,\Im\rangle$ is said to be compact if every $\Im$-open covering of $S$ has a finite subcovering. ♦

**Theorem 1.2-1**: The topological space $\langle\Re,\mathsf{U}\rangle$ is not compact. Therefore, no open interval on $\Re$ is compact in its relativized $\mathsf{U}$-topology.

**Proof of Theorem 1.2-1**: Let $\Sigma = \{(-k,k) : k \in \mathbf{N}\}$. Then, $\Sigma$ is a $\cup$-open covering of $\Re$ because each member of $\Sigma$ is an open interval in $\Re$, and $\Re \subset \bigcup_{k \in \mathbf{N}} (-k,k)$. If $x \in \Re$, then $\exists\, n_x \in \mathbf{N}$ such that $n_x > |x| \Rightarrow x \in (-n_x, n_x)$. So, $x \in \bigcup_{n \in \mathbf{N}} (-n,n)$. Now, let $(-n_1,n_1),(-n_2,n_2),\cdots,(-n_k,n_k)$ be any finite collection of members of $\Sigma$ and let $n^* = \max(n_1,n_1,\cdots,n_k)$. Then, $n^* \notin \bigcup_{k \in \mathbf{N}} (-n_k, n_k)$. Therefore, there is no finite collection of members of $\Sigma$ which is a covering of $\Re$. So, $\langle \Re, \cup \rangle$ is not compact. The second assertion follows from homeomorphism between $\langle \Re, \cup \rangle$ and $\langle \mathbf{I}, \cup \rangle$ where $\mathbf{I}$ is an interval in $\Re$. ♦

**Definition 1.2-15**: A mapping $f : X \to \Re$ is bounded if the range of $f[X]$ is a bounded subset of $\Re$. ♦

Next we present the following important results without proof.

**Result 1.2-4**: If $\langle X, \Im_X \rangle$ is a compact subspace of a Hausdorff space $\langle \Omega, \Im \rangle$, then $X$ is $\Im$-closed. ♦

**Result 1.2-5**: If $\langle \Omega, \Im \rangle$ is compact and $X$ is a $\Im$-closed subset of $X$ , then $\langle X, \Im_X \rangle$ is compact. ♦

**Result 1.2-6**: (Heine-Borel) For $X \subset \Re$, $\langle X, \mathbf{\cup}_X \rangle$ is compact iff $X$ is bounded and $\mathbf{\cup}$-closed.♦

**Result 1.2-7**: A continuous image of a compact space is compact. That is, for two topological spaces $\langle X, \Im \rangle$ and $\langle Y, \vartheta \rangle$, if $f : X \to Y$ is $\Im - \vartheta$-continuous, compactness of $\langle X, \Im \rangle$ implies compactness of $\langle Y, \vartheta \rangle$. ♦

**Result 1.2-8**: Let $\langle X, \Im \rangle$ be a compact space and let $\langle Y, \vartheta \rangle$ be a Hausdorff space. If $f : X \to Y$ is $\Im - \vartheta$-continuous and surjective, then $\mathrm{f}$ is a homeomorphism. ♦

**Result 1.2-9**: Let $\langle X, \Im \rangle$ be a compact space. If $f : X \to \Re$ is $\Im - \cup$-continuous, then $\mathrm{f}(\cdot)$ is bounded. ♦

**Result 1.2-10**: (Bolzano-Weierstrass): Every bounded infinite subset of $\Re$ has a cluster point. ♦

**Result 1.2-11**: Let $X \subseteq \Re$ be bounded and $\cup$-closed. If $f : X \to \Re$ is $\cup_{\mathrm{X}} - \cup$-continuous, then $f$ is bounded. ♦

## Total Boundedness and Approximation

**Definition 1.2-16**: Let $E$ be a set in a metric space $\langle S, d \rangle$. Given $\varepsilon > 0$, $E_\varepsilon \subset E$ is an $\varepsilon -$ **net** of $E$ if :
(i) $E_\varepsilon$ is a finite set; and (ii) $\forall x \in E,$ there exists $y \in E_\varepsilon$ such that $d(x,y) < \varepsilon.$
A set $E$ in $\langle S, d \rangle$ is **totally bounded** if $\forall \varepsilon > 0,$ there exists an $\varepsilon -$ net in $E.$ ♦

**Remark 1.2-3**: Total boundedness implies boundedness. The converse is true for all finite-dimensional spaces but, in general, it is not true for infinite-dimensional spaces. ♦

**Remark 1.2-3**: Every finite set in a metric space is bounded and hence it is totally bounded. ♦

**Example 1.2-8**: Consider the closed ball $\widetilde{B}_1(\underline{0}) = \{x \in \ell_2 : \|x\|_{\ell_2} \le 1\}$ where the distance function is defined as:

$$d(x,y) = \|x - y\|_{\ell_2} \equiv \sqrt{\sum_{k=1}^{\infty} |x_k - y_k|^2} \;\; \forall x, y \in \ell_2$$

The set $\widetilde{B}_1(\underline{0})$ is bounded because $d(x,y) \le 2 \; \forall x, y$ but $\widetilde{B}_1(\underline{0})$ is not totally bounded as seen below.

Let us construct $E = \{e^k : k \in \mathbf{N}\}$ where $e^k$ is the sequence of all 0's except '1' as the $k^{th}$ element of the sequence. Clearly, $d(e^k, e^\ell) = \sqrt{2}\delta_\ell^k$. If an $\varepsilon -$ net $E_{1/2}$ exists for $\varepsilon = 1/2$, then $E_{1/2}$ must be a finite subset in $E$. But since the closed balls $\widetilde{B}_{1/2}(e^k)$ and $\widetilde{B}_{1/2}(e^\ell)$ are disjoint for all $k \ne \ell$, $E_{1/2}$ must contains a point within a distance $\frac{1}{2}$ of each $e^k$. Since there are countably many $e^k$'s, $E_{1/2}$ cannot be a finite set. However, notice that this violation of finite cardinality would not have occurred in a finite-dimensional space.

Next we present the following important results without proof.

**Result 1.2-12**: Let $E \subseteq X$ in a metric space $\langle X, d \rangle$. If E has an $\varepsilon -$net for some $\varepsilon > 0$, then $E$ is bounded. ♦

**Result 1.2-13**: Let $\langle X, d \rangle$ be a totally bounded metric space. Then, $X$ is separable. ♦

**Result 1.2-14**: A bounded set $E \subseteq \ell_2$ is totally bounded iff $\forall \varepsilon > 0 \; \exists \; n(\varepsilon) \in \mathbb{N}$ such that

$$\sum_{k=n}^{\infty} |x_k|^2 < \varepsilon \;\; \forall x \in E$$ ♦

**Result 1.2-15**: Let $E \subseteq X$ in a metric space $\langle X, d \rangle$. Then, the following statements are equivalent:

(i) The closure $\overline{E}$ is sequentially compact.

(ii) Every sequence in E has a subsequence that converges in $X$. ♦

**Result 1.2-16**: Every sequentially compact set in a metric space is closed. ♦

**Result 1.2-17**: Every sequentially compact metric space is complete. ♦

**Result 1.2-18**: Every sequentially compact metric space is totally bounded. ♦

**Result 1.2-19**: A metric space is sequentially compact iff it is totally bounded and complete. ♦