Chapter Three: Normed Vector Spaces

The concepts of convergence and continuity in models of physical processes have been discussed in Chapter One on metric spaces and topological spaces. The notion of linearity in algebraic structures of vector spaces has been presented in Chapter Two. The treatment of analysis in these two chapters was intentionally made independent of each other. In other words, Chapter Two could have been covered before Chapter One. The purpose of this Chapter Three is to present a special class of vector spaces that would combine the two notions: continuity and linearity. This would ultimately lead to continuous linear transformations that include linear functionals and linear operators as they are extensively used in Engineering, Physics, and other disciplines.

The underlying vector space, presented in this chapter, synergistically combines the topological structure of metric spaces and the algebraic structure of vector spaces to yield normed vector spaces. The combination is made in such a way that these two structures, namely, topological and algebraic, are compatible in the sense that the addition and multiplication operations are continuous and the resulting metric is readily related to the scalar field of the vector space. This chapter should be read along with Chapter 5 Part A of Naylor & Sell. Specifically, both solved examples and exercises in Naylor & Sell are very useful.

1 Basic Concepts

Let \((V, \odot)\) be a vector space over a (complete) field \(\mathbb{F}\), where we choose the field \(\mathbb{F} = (F, +, \cdot)\) to be either \(\mathbb{R}\) or \(\mathbb{C}\). [Note: Both \(\mathbb{R}\) and \(\mathbb{C}\) are complete fields. The field \(\mathbb{Q}\) is not considered because it is not complete in the usual metric.] We wish to introduce a distance \(d : V \times V \to [0, \infty)\) between a pair of points in \(V\). This will allow us to define various mathematical entities such as limits, convergent sequences and series, and continuous mappings. Since \(V\) has the algebraic structure of a vector space, the distance that we will define must be consistent with this algebraic structure. To this end, we will require the following properties:

**Property 01** Translation invariance: \(d(x, y) = d(x \odot z, y \odot z)\) \(\forall x, y, z \in V\).

**Property 2** Positive homogeneity: \(d(\lambda x, \lambda y) = |\lambda|d(x, y)\) \(\forall x, y \in V\) and \(\forall \lambda \in \mathbb{F}\).

**Property 3** Convexity: Every open ball \(B(x_0, r) \triangleq \{x \in V : d(x, x_0) < r\}\) is a convex set.
The invariance under translation implies that the distance $d$ is entirely determined if we specify the function $x \mapsto \|x\| = d(x, 0)$, i.e., the distance of a point $x$ from the origin. This is what we call the norm of a vector, which is formally defined below.

**Definition 1.1.** (Norm) Let $(V, \oplus)$ be a vector space over a (complete) field $F$. Then, a function $\|\cdot\| : V \to [0, \infty)$ is called a norm if the following conditions hold $\forall x, y \in V \forall \alpha \in F$:

- **Strict positivity:** $\|x\| = 0$ iff $x = 0$
- **Homogeneity:** $\|\alpha x\| = |\alpha| \|x\|
- **Triangular Inequality:** $\|x \oplus y\| \leq \|x\| + \|y\|

A vector space with a norm is called a normed vector space and is denoted as the pair $(V, \|\cdot\|)$.

**Remark 1.1.** If all of the above conditions of a norm except $(\|x\| = 0 \Rightarrow x = 0_V$ is satisfied, then we introduce the notion of seminorm $p(\cdot)$ by setting $d(x, y) = p(x - y)$, where it is not guaranteed that we obtain a distance on the vector space. Indeed, we may have $d(x, y) = p(x - y) = 0$ even if $x \neq y$. There are, however, cases such as the $L_p$-space where one may convert a seminorm into a norm by constructing equivalence classes of seminorms. See, for example, p. 66 of Real and Complex Analysis by Rudin.

**Definition 1.2.** (Separating seminorm) A sequence $\{p_n\}$ of seminorms is called separating if, for every $x \in V \setminus \{0\}$, there exists at least one index $k$ such that $p_k(x) > 0$.

**Lemma 1.1.** (Distance generated by seminorms) Let $\{p_n\}$ be a sequence of separating seminorms on a vector space $V$. Then, a distance on $V$ can be defined as:

$$
\begin{align*}
d(x, y) & \equiv \sum_{k=1}^{\infty} 2^{-k} \frac{p_k(x - y)}{1 + p_k(x - y)} \\
& \leq \sum_{k=1}^{\infty} 2^{-k} \left( \frac{p_k(x - y)}{1 + p_k(x - y)} + \frac{p_k(y - z)}{1 + p_k(y - z)} \right)
\end{align*}
$$

Proof. The identities $d(x, x) = 0$ and $d(x, y) = d(y, x)$ are immediate consequences of strict positivity and homogeneity in Definition 1.1. The assertion that the sequence $\{p_n\}$ is separating guarantees that $d(x, y) > 0$ if $x \neq y$.

Now, what remains to complete the proof is triangular inequality. We observe that if the real numbers $a, b, c \geq 0$ and $c \leq (a + b)$, then it follows that

$$
\frac{c}{1 + c} \leq \frac{a + b}{1 + a + b} \leq \left( \frac{a}{1 + a} + \frac{b}{1 + b} \right)
$$

because the function $s \mapsto \frac{s}{1 + s}$ is increasing and concave down. By using the property of triangular inequality in Definition 1.1 with $a \equiv p_k(x - y)$, $b \equiv p_k(y - z)$ and $c \equiv p_k(z - x)$, it follows that

$$
\begin{align*}
d(x, z) & = \sum_{k=1}^{\infty} 2^{-k} \frac{p_k(x - z)}{1 + p_k(x - z)} \\
& \leq \sum_{k=1}^{\infty} 2^{-k} \left( \frac{p_k(x - y)}{1 + p_k(x - y)} + \frac{p_k(y - z)}{1 + p_k(y - z)} \right)
\end{align*}
$$

i.e., $d(x, z) \leq d(x, y) + d(y, z)$. Hence, $d(\cdot, \cdot)$ is indeed a distance. \(\square\)
If the vector space $V$ with the distance in Eq. (1) is a complete metric space, then $V$ is called a Frechét space.

Example 1.1. Examples of Normed Spaces:

1. Let $V = F^n$ and $p \in [1, \infty)$. Then, $\|x\|_p \triangleq \left(\sum_{k=1}^n |x_k|^p\right)^{1/p}$ is a norm, where the vector $x \equiv [x_1, x_2, \ldots, x_n]^T$.

2. Let $V = F^n$. Then, the sup norm $\|\cdot\|_\infty$ is defined as: $\|x\|_\infty \triangleq \max_{k \in \{1,2,\ldots,n\}} |x_k|$.

Remark 1.2. It follows from the triangular inequality of a norm in $V$ that $|\|x\| - \|y\|| \leq \|x-y\| \forall x, y \in V$.

Remark 1.3. A norm can be considered to be a metric of the vector space by setting $d(x, y) = \|x-y\|$. However, a metric may not satisfy the homogeneity condition of a norm because there is no notion of a (scalar) field related to a metric space. So, every normed vector space is a metric space but the converse is not true, in general.

Proposition 1.1. The norm is a uniformly continuous function.

Proof. Choose $\delta(\epsilon, x) = \epsilon \forall x$. Then, $\forall \epsilon > 0 \forall x, y \in V$, $\|x\| - \|y\| \leq \|x-y\| < \epsilon \Rightarrow \|x\| - \|y\| < \epsilon$. \qed

Definition 1.3. (Lipschitz Continuity) Let $V$ and $W$ be two normed vector spaces over the same field $F$. A function $f : V \rightarrow W$ is called Lipschitz continuous if there exists a constant $c \in (0, \infty)$, called the Lipschitz constant, such that $\|f(x) - f(y)\|_W \leq c\|x-y\|_V \forall x, y \in V$.

Proposition 1.2. Lipschitz continuity $\Rightarrow$ uniform continuity.

Proof. Choose $\delta(\epsilon, x) = \frac{\epsilon}{c} \forall x$. Then, $\forall \epsilon > 0 \forall x, y \in V$, $\|x\| - \|y\| \leq \|x-y\| < \epsilon \Rightarrow c\|x\| - \|y\| \leq \epsilon$. \qed

Remark 1.4. Uniform continuity $\Rightarrow$ Lipschitz continuity, in general.

Definition 1.4. (Contraction Mapping) Let $V$ a normed vector space over the field $F$. A mapping $f : V \rightarrow V$ is called a contraction if there exists a constant $\rho \in (0, 1)$, such that $\|f(x) - f(y)\|_V \leq \rho\|x-y\|_V \forall x, y \in V$.

Definition 1.5. (Norm Equivalence) Let $V$ a normed vector space over the field $F$. Then, two norms $\|\cdot\|_a$ and $\|\cdot\|_b$ on $V$ are said to be equivalent if there exists $\alpha_m, \alpha_M \in (0, \infty)$ such that $0 < \alpha_m \|x\|_a \leq \|x\|_b \leq \alpha_M \|x\|_a \forall x \in V \setminus \{0\}$.

Alternatively, $0 < \frac{1}{\alpha_M} \|x\|_b \leq \|x\|_a \leq \frac{1}{\alpha_m} \|x\|_b \forall x \in V \setminus \{0\}$.

Remark 1.5. From a topological perspective, the implication of norm equivalence in Definition 1.5 is that the two norms, considered as respective metrics in the underlying vector space, generate the same topology.
1.1 Finite-dimensional Normed Spaces

This section deals with finite-dimensional normed spaces. The set of finite-dimensional normed spaces is a proper subset of the set of all normed spaces that include infinite-dimensional normed spaces. Nevertheless finite-dimensional normed spaces are useful for analysis of many engineering systems. To this end we present below several important theorems and lemmas.

**Theorem 1.1.** *(Norm Equivalence in Finite-dimensional Spaces)* On a finite-dimensional vector space, all norms are equivalent.

*Proof.* Let $V$ be an $n$-dimensional vector space over a field $F$, where $n \in \mathbb{N}$. Since $V$ is isomorphic to $F^n$, it suffices to show norm equivalence on $F^n$.

Let a bounded set in $F^n$ be defined as: $S_{a}^{n-1} \triangleq \{ x \in F^n : \| x \|_a = 1 \}$. It follows by continuity of a norm and $S_{a}^{n-1}$ being the inverse image of the closed set $\{ 1 \}$ (in the usual topology on $\mathbb{R}$) that $S_{a}^{n-1}$ is closed in $(F^n, \| \cdot \|_a)$. Notice that $0 \notin S_{a}^{n-1}$.

Let $\| \cdot \|_b$ be an arbitrary norm in $F^n$. Thus, $\| \cdot \|_b$ is a continuous function with $z \mapsto \| z \|_b$, which is restricted to the closed and bounded set $S_{a}^{n-1}$ (which is compact in the finite-dimensional space $F^n$). Therefore, by following the boundedness theorem on real-valued continuous functions on compact sets (see Theorem 4.16 (page 89) of Principles of Mathematical Analysis by W. Rudin), $\| \cdot \|_b$ reaches its minimum and maximum values at $z_m$ and $z_M$, respectively, where $0 < \| z_m \|_b \leq \| z \|_b \leq \| z_M \|_b \forall z \in S_{a}^{n-1}$.

Let $x \in F^n \setminus \{ 0 \}$. Then, $z \triangleq \frac{x}{\| x \|_a} \in S_{a}^{n-1}$.

Therefore, $0 < \| z_m \|_b \| x \|_a \leq \| x \|_b \leq \| z_M \|_b \| x \|_a \forall x \in S_{a}^{n-1}$.

\[
\| \sum_{j=1}^{n} \alpha_j v_j \| \geq c \sum_{j=1}^{n} | \alpha_j |
\]

*Proof.* Let us denote $s = \sum_{j=1}^{n} | \alpha_j |$. If $s = 0$, then each $\alpha_j$ is zero and the proof follows trivially. Letting $s > 0$ (i.e., ensuring that not all $\alpha_j$’s are zero) and setting $\beta_j = \alpha_j / s$, it suffices to show that

\[
\| \sum_{j=1}^{n} \beta_j v_j \| \geq c \left( \text{where } \sum_{j=1}^{n} | \beta_j | = 1 \right)
\]

We establish the above assertion by contradiction. Suppose it is false. Then, there exists a sequence $\{ y^k \}$ of vectors such that

\[
\| y^k \| \to 0
\]

Let $\beta_k = \frac{\alpha_j}{s}$ for each $\alpha_j$ and $\| y^k \| = \beta_k$. Then, for any $\epsilon > 0$, there exists an $N$ such that $\| y^k \| < \epsilon$ for $k > N$. Thus, $\| \alpha_j \| < \epsilon s$ for $k > N$.

Next we present an alternative proof of Theorem 1.5 for equivalence of all norms in a finite-dimensional vector space, which requires the following lemma.

**Lemma 1.2.** *(Linear combination in Normed Spaces)* Let $V$ be a vector space over a (complete) field $F$ that is either $\mathbb{R}$ or $\mathbb{C}$. Let $\{ v^1, \ldots, v^n \}$ be a linearly independent set of vectors in the normed space $(V, \| \cdot \|)$. (Note that the space $V$ could be either finite-dimensional or infinite-dimensional). Then, there exists a real number $c \in (0, \infty)$ such that, for an arbitrary choice of scalars $\alpha_j \in F, j = 1, \ldots, n$ with at least one $\alpha_j \neq 0$, the following inequality holds:

\[
\| \sum_{j=1}^{n} \alpha_j v_j \| \geq c \sum_{j=1}^{n} | \alpha_j |
\]

*Proof.* Let us denote $s = \sum_{j=1}^{n} | \alpha_j |$. If $s = 0$, then each $\alpha_j$ is zero and the proof follows trivially. Letting $s > 0$ (i.e., ensuring that not all $\alpha_j$’s are zero) and setting $\beta_j = \alpha_j / s$, it suffices to show that

\[
\| \sum_{j=1}^{n} \beta_j v_j \| \geq c \left( \text{where } \sum_{j=1}^{n} | \beta_j | = 1 \right)
\]

We establish the above assertion by contradiction. Suppose it is false. Then, there exists a sequence $\{ y^k \}$ of vectors such that
\[ y^k = \left\| \sum_{j=1}^{n} \beta_j^k v^j \right\| \left( \text{where } \sum_{j=1}^{n} |\beta_j^k| = 1 \right) \]

such that \( \|y^k\| \to 0 \) as \( k \to \infty \). Having \( \sum_{j=1}^{n} |\beta_j^k| = 1 \), it follows that \( |\beta_j^k| \leq 1 \Rightarrow \{\beta_j^k\} \) is bounded for each \( j = 1, \cdots, n \). Consequently, by Bolzano-Weierstrass theorem, it follows that the bounded sequence of scalars \( \{\beta_j^k\} \) has a convergent subsequence; let this subsequence converge to \( \beta_1 \). Let the corresponding subsequence of the sequence of vectors \( \{y^k\} \) be denoted as \( \{y_1^k\} \). By the same argument, the sequence of vectors \( \{y_1^k\} \) has a subsequence \( \{y_2^k\} \) for which the corresponding subsequence of scalars \( \{\beta_2^k\} \) converges to \( \beta_2 \). Continuing in this way for \( n \) steps, we obtain a subsequence of vectors \( \{y_n^k\} \) whose terms are of the form

\[ y_n^k = \sum_{j=1}^{n} \gamma_j^k v^j \left( \text{where } \sum_{j=1}^{n} |\gamma_j^k| = 1 \right) \]

and the scalars \( \gamma_j^k \to \beta_j \) as \( k \to \infty \). Hence, as \( k \to \infty \) \( y_n^k \to y = \sum_{j=1}^{n} \beta_j v^j \)

(\( \text{where } \sum_{j=1}^{n} |\beta_j| = 1 \)) so that not all \( \beta_j \)'s are zero.

Since \( \{v^1, \cdots, v^n\} \) is a linearly independent set of vectors, it is ensured that \( y \neq 0 \). On the other hand, by continuity of the norm, \( y_n^k \to y \) implies \( \|y_n^k\| \to \|y\| \). Since \( \|y^k\| \to 0 \) by the original assumption and \( \{y_1^k\} \) is a subsequence of \( \{y^k\} \), it follows that \( \|y_1^k\| \to 0 \). Hence, \( \|y\| \to 0 \Rightarrow y = 0 \). This is a contradiction of the original assumption that \( y \neq 0 \). The proof of the lemma is complete.

By setting \( \alpha_m \triangleq \|z_m\|_b \) and \( \alpha_M \triangleq \|z_M\|_b \), equivalence of the norms \( \|\bullet\|_a \) and \( \|\bullet\|_b \) is established.

\( \square \)

Now, by using Lemma 1.2, we present the alternative proof of Theorem 1.5.

**Proof.** Let \( \{v^1, \cdots, v^n\} \) be a basis for \( \mathbb{F}^n \). Then, a vector \( x \in \mathbb{F}^n \setminus \{0\} \) is expressed as \( x = \sum_{j=1}^{n} \alpha_j v^j \), where \( \alpha_j \in \mathbb{F} \).

Given a norm \( \|\bullet\|_a \) on \( \mathbb{F}^n \), there exists a real \( c_a \in (0, \infty) \) such that \( \|x\|_a = c_a \sum_{j=1}^{n} |\alpha_j| \) by Lemma 1.2.

Next we consider the norm \( \|\bullet\|_b \) and apply the triangular inequality to obtain

\[ \|x\|_b \leq k_b \sum_{j=1}^{n} |\alpha_j|, \]

where \( k_b \triangleq \max\{\|v^1\|_b, \cdots, \|v^n\|_b\} \), which implies that \( \|x\|_b \leq k_b \frac{c_a}{c_b} \|x\|_a \)

By repeating the process with the norms \( \|\bullet\|_a \) and \( \|\bullet\|_b \) reversed, we obtain

\[ \|x\|_a \leq k_a \frac{c_b}{c_a} \|x\|_b \Rightarrow \|x\|_b \geq k_a \frac{c_a}{c_b} \|x\|_a . \]

By setting \( \alpha_m \triangleq \frac{k_a}{c_a} \) and \( \alpha_M \triangleq \frac{k_b}{c_b} \), equivalence of the norms \( \|\bullet\|_a \) and \( \|\bullet\|_b \) is established. \( \square \)

## 2 Summable Sequences and Integrable Functions

**Definition 2.1.** (Space of summable sequences \( \ell_p \) and \( \ell_\infty \)) Let \( x \triangleq \{x_k\} \) and \( y \triangleq \{y_k\} \) be two sequences in a normed space \((V, \|\bullet\|)\) over the field \( \mathbb{F} \), where the vector addition and scalar multiplication are denoted as: \( z = x \oplus y \Rightarrow z_k = x_k + y_k \)

and \( w = \alpha \otimes x \Rightarrow w_k = \alpha x_k \forall x, y, z \in V \forall \alpha \in \mathbb{F} \).

A subspace of the normed space of summable sequences is defined to be \( \ell_p \) for a given \( p \in [1, \infty) \) if \( \sum_{k=1}^{\infty} |x_k|^p < \infty \) for all \( x \in \ell_p \); and the resulting norm is denoted
as: \( \|x\|_{L_p} \triangleq \left( \sum_{k=1}^{\infty} |x_k|^p \right)^{\frac{1}{p}} \).

A subspace of the normed space of summable sequences is defined to be \( \ell_\infty \) if \( \sup_{k \in \mathbb{N}} |x_k| < \infty \) for all \( x \in \ell_\infty \); and the resulting norm is denoted as: \( \|x\|_{\ell_\infty} \triangleq \sup_{k \in \mathbb{N}} |x_k| \).

**Definition 2.2.** (Space \( L_p \) of Lebesgue-integrable functions) In a non-measure-theoretic setting, the space \( L_p \) of Lebesgue-integrable functions on a set \( T \subseteq \mathbb{R} \) is defined as:

\[
L_p(T) \triangleq \left\{ x : \int_T \, dt \, |x(t)|^p < \infty \right\} \quad \text{and} \quad \|x\|_{L_p} \triangleq \left( \int_T \, dt \, |x(t)|^p \right)^{\frac{1}{p}} \quad \text{for} \quad p \in [1, \infty).
\]

In a measure-theoretic setting, the space \( L_p \) of Lebesgue-integrable functions in the measure space \((\mathbb{R}, \mathcal{B}, \mu)\) is defined as:

\[
L_p(\mu) \triangleq \left\{ x : \int_\mathbb{R} \, d\mu(t) \, |x(t)|^p < \infty \right\} \quad \text{and} \quad \|x\|_{L_p} \triangleq \left( \int_\mathbb{R} \, d\mu(t) \, |x(t)|^p < \infty \right)^{\frac{1}{p}} \quad \text{for} \quad p \in [1, \infty).
\]

**Remark 2.1.** The measure space is not restricted to \((\mathbb{R}, \mathcal{B}, \mu)\). For example, the space \( \mathbb{R} \) could be replaced by \( \mathbb{R}^n \) and accordingly the \( \sigma \)-algebra \( \mathcal{B} \) by \( \mathcal{B}^n \) and the Lebesgue measure \( \mu \) by an appropriate product measure \( \pi \).

**Definition 2.3.** (Equivalence Class) The equivalence class generated by a bounded function \( x \) in \((\mathbb{R}, \mathcal{B}, \mu)\) is defined as: \( \mathcal{E}(x) \triangleq \left\{ y : y = x \ \mu\text{-almost everywhere (ae)} \right\} \), where \( y = x \ \mu\text{-ae} \Rightarrow y(t) \neq x(t) \) only on a set of zero \( \mu \)-measure.

**Definition 2.4.** (Essential Supremum) Let \( \mathcal{E}(x) \) be the equivalence class generated by a bounded function \( x \) in \((\mathbb{R}, \mathcal{B}, \mu)\). Then, the essential supremum of the function \( x \) is defined as: \( \text{ess sup} \ x \triangleq \inf_{\tilde{\varepsilon} \in \mathcal{E}(x)} \sup_{t \in \mathbb{R}} |\tilde{x}(t)| \).

**Definition 2.5.** (Space \( L_\infty \) of essentially bounded functions) Let \( \mathcal{E}(\bullet) \) be the equivalence class in \((\mathbb{R}, \mathcal{B}, \mu)\). Then, the space of essentially bounded functions is defined as: \( L_\infty(\mu) \triangleq \left\{ x : \text{ess sup} \ x < \infty \right\} \) and the associated norm is defined as:

\[
\|x\|_{L_\infty} \triangleq \text{ess sup} \ x.
\]

**Remark 2.2.** To make \( \|x\|_{L_p} \) and \( \|x\|_{L_\infty} \) valid norms (instead of being semi-norms) in a measure space \((\mathbb{R}, \mathcal{B}, \mu)\), a necessary condition is that \( \|x\| = 0 \) if and only if \( x = 0 \) for both \( L_p \) and \( L_\infty \). This equality is true in the \( \mu \)-almost everywhere sense.

In essence, the vectors in both \( L_p \) and \( L_\infty \) are equivalence classes of functions, *not* individual functions. In other words, a quotient space is constructed from the vector space of individual functions under the equivalence relation \( \mathcal{E}(\bullet) \).

**Lemma 2.1.** If \( f \in L_1(\mu) \), then

\[
\left| \int d\mu(t) f(t) \right| \leq \int d\mu(t) |f(t)|
\]

**Proof.** Let \( z = \int d\mu(t) f(t) \) which is, in general, a complex number, i.e., \( z \in \mathbb{C} \). There exists \( \alpha \in \mathbb{C} \) with |\( \alpha \)| = 1 such that \( \alpha z = |z| \). Let \( u \) be the real part of \( \alpha f \). Then, \( u \leq |\alpha f| = |f| \). Hence,

\[
\left| \int d\mu(t) f(t) \right| = |\alpha| \int d\mu(t) f(t) = \int d\mu(t) \alpha f(t) = \int d\mu(t) u(t) \leq \int d\mu(t) |f(t)|
\]

The third of the above equalities holds because the preceding equalities show that \( \int d\mu(t) \alpha f(t) \) is real. \( \square \)
2.1 Inequalities in $\ell_p$ and $L_p$ Spaces for $p \in [1, \infty]$  

The goal of this subsection is to introduce Minkowski inequality that is the triangular inequality in $\ell_p$ and $L_p$ Spaces for $p \in [1, \infty]$. In this context, we establish another inequality, known as Hölder inequality, which is extensively used in real and functional analysis as well as bears significant importance in engineering analysis.

\[ \text{Figure 1: Geometric interpretation of scalar inequality} \]

**Lemma 2.2.** (Scalar Inequality) Given $p \in (1, \infty)$, let $q \in (1, \infty)$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. (Note that $p$ and $q$ are called conjugate exponents.) Let $a \in [0, \infty)$ and $a \in [0, \infty)$. Then, $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$.

**Proof.** It follows from the given conditions on $p$ and $q$ that:

\[ 1 = \frac{p + q}{pq} \Rightarrow pq = p + q \Rightarrow (p - 1)(q - 1) = 1 \]

In the $(\xi, \eta)$-plane, consider the curve $\eta = \xi^{p-1}$, or equivalently, $\xi = \eta^{q-1}$. Let

\[ A_1 \triangleq \int_0^a d\xi \xi^{p-1} = \frac{a^p}{p} \quad \text{and} \quad A_2 \triangleq \int_0^b d\eta \eta^{q-1} = \frac{b^q}{q} \]

By interpreting $A_1$ and $A_2$ as areas, as shown in Figure 1, it follows that $ab \leq A_1 + A_2$. 

**Theorem 2.1.** (Hölder Inequality) Given $p \in (1, \infty)$, let $q \in (1, \infty)$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Then, the following inequalities hold.

1. Finite Sums:
\[
\left| \sum_{j=1}^{n} x_j y_j \right| \leq \sum_{j=1}^{n} |x_j y_j| \leq \left( \sum_{j=1}^{n} |x_j|^p \right)^{\frac{1}{p}} \left( \sum_{j=1}^{n} |y_j|^q \right)^{\frac{1}{q}}
\]

2. Infinite Sums: Given $x \in \ell_p(\mu)$, i.e., $\sum_{j=1}^{\infty} |x_j|^p < \infty$ and $y \in \ell_q(\mu)$, i.e., $\sum_{j=1}^{\infty} |y_j|^q < \infty$,
\[
\left| \sum_{j=1}^{\infty} x_j y_j \right| \leq \sum_{j=1}^{\infty} |x_j y_j| \leq \left( \sum_{j=1}^{\infty} |x_j|^p \right)^{\frac{1}{p}} \left( \sum_{j=1}^{\infty} |y_j|^q \right)^{\frac{1}{q}}
\]
3. Integrals over $\Omega \subseteq \mathbb{R}$: Given $x \in L_p(\mu)$, i.e., $\int_{t \in \Omega} d\mu(t)|x(t)|^p < \infty$ and $y \in L_q(\mu)$, i.e., $\int_{t \in \Omega} d\mu(t)|y(t)|^q < \infty$,

$$\left| \int_{t \in \Omega} d\mu(t)x(t)y(t) \right| \leq \left( \int_{t \in \Omega} d\mu(t)|x(t)|^p \right)^{\frac{1}{p}} \left( \int_{t \in \Omega} d\mu(t)|y(t)|^q \right)^{\frac{1}{q}}$$

**Proof.** See Naylor & Sell p. 550, where Hölder inequality is proved by making use of the scalar inequality in Lemma 2.2.

**Remark 2.3.** Hölder inequality is extendable to $p = 1$ and $p = \infty$, which implies that and $q = \infty$ and $q = 1$, respectively, by using the relation $\frac{1}{p} + \frac{1}{q} = 1$ on the extended real line $\overline{\mathbb{R}}$, as explained below.

1. **Finite Sums:**

$$\sum_{j=1}^{n} |x_j y_j| \leq \left( \sum_{j=1}^{n} |x_j| \right) \left( \max_j |y_j| \right)$$

2. **Infinite Sums:** Given $x \in \ell_1(\mu)$, i.e., $\sum_{j=1}^{\infty} |x_j| < \infty$ and $y \in \ell_\infty(\mu)$, i.e., $\sup_j |y_j| < \infty$,

$$\sum_{j=1}^{\infty} |x_j y_j| \leq \left( \sum_{j=1}^{\infty} |x_j| \right) \left( \sup_j |y_j| \right)$$

3. **Integrals over $\Omega \subseteq \mathbb{R}$:** Given $x \in L_1(\mu)$, i.e., $\int_{t \in \Omega} d\mu(t)|x(t)| < \infty$ and $y \in L_\infty(\mu)$, i.e., $\text{ess sup}_{t \in \Omega} |y(t)| < \infty$,

$$\int_{t \in \Omega} d\mu(t)|x(t)y(t)| \leq \left( \int_{t \in \Omega} d\mu(t)|x(t)| \right) \left( \text{ess sup}_{t \in \Omega} |y(t)| \right)$$

**Remark 2.4.** In Theorem 2.1, for the special case $p = 2$ that mandates $q = 2$ by the relation $\frac{1}{p} + \frac{1}{q} = 1$, Hölder inequality is called the Schwarz inequality.

**Theorem 2.2.** (Minkowski Inequality) Let $p \in [1, \infty)$. Then, the following inequalities hold.

1. **Finite Sums:**

$$\left( \sum_{j=1}^{n} |x_j \pm y_j|^p \right)^{\frac{1}{p}} \leq \left( \sum_{j=1}^{n} |x_j|^p \right)^{\frac{1}{p}} + \left( \sum_{j=1}^{n} |y_j|^p \right)^{\frac{1}{p}}$$

2. **Infinite Sums:** Given $x \in \ell_p(\mu)$, i.e., $\sum_{j=1}^{\infty} |x_j|^p < \infty$ and $y \in \ell_p(\mu)$, i.e., $\sum_{j=1}^{\infty} |y_j|^p < \infty$,

$$\left( \sum_{j=1}^{\infty} |x_j \pm y_j|^p \right)^{\frac{1}{p}} \leq \left( \sum_{j=1}^{\infty} |x_j|^p \right)^{\frac{1}{p}} + \left( \sum_{j=1}^{\infty} |y_j|^p \right)^{\frac{1}{p}}$$

3. **Integrals over $\Omega \subseteq \mathbb{R}$:** Given $x \in L_p(\mu)$, i.e., $\int_{t \in \Omega} d\mu(t)|x(t)|^p < \infty$ and $y \in L_p(\mu)$, i.e., $\int_{t \in \Omega} d\mu(t)|y(t)|^p < \infty$,

$$\left( \int_{t \in \Omega} d\mu(t)|x(t) \pm y(t)|^p \right)^{\frac{1}{p}} \leq \left( \int_{t \in \Omega} d\mu(t)|x(t)|^p \right)^{\frac{1}{p}} + \left( \int_{t \in \Omega} d\mu(t)|y(t)|^p \right)^{\frac{1}{p}}$$
Proof. See Naylor & Sell pp. 550-551, where Minkowski inequality is proved by making use of Hölder inequality in Theorem 2.1. \qed

**Remark 2.5.** Minkowski inequality is extendable to \( p = \infty \) on the extended real line \( \mathbb{R} \), as explained below.

1. **Finite Sums:**
   \[
   \max_j |x_j \pm y_j| \leq \left( \max_j |x_j| + \max_j |y_j| \right)
   \]

2. **Infinite Sums:** Given \( x \in \ell_{\infty}(\mu) \), i.e., \( \sup_j |x_j| < \infty \) and \( y \in \ell_{\infty}(\mu) \), i.e., \( \sup_j |y_j| < \infty \),
   \[
   \sup_j |x_j \pm y_j| \leq \left( \sup_j |x_j| + \sup_j |y_j| \right)
   \]

3. **Integrals over \( \Omega \subseteq \mathbb{R} \):** Given \( x \in L_{\infty}(\mu) \), i.e., \( \ess\sup_{t \in \Omega} |x(t)| < \infty \) and \( y \in L_{\infty}(\mu) \), i.e., \( \ess\sup_{t \in \Omega} |y(t)| < \infty \),
   \[
   \ess\sup_{t \in \Omega} |x(t) \pm y(t)| \leq \left( \ess\sup_{t \in \Omega} |x(t)| \right) + \left( \ess\sup_{t \in \Omega} |y(t)| \right)
   \]

**Remark 2.6.** Theorem 2.2 establishes the triangular inequality in \( \ell_p \) and \( L_p \), as well as spaces \( \ell_{\infty} \) and \( L_{\infty} \) spaces. Therefore, it is established that these spaces are normed vector spaces. In the next section, we show that these spaces are complete under the respective norms (used as metrics), i.e., they are Banach spaces.

**Theorem 2.3.** *(Inclusion Theorem 1)* Given \( 1 < p < q < \infty \), the following condition holds: \( \ell_1 \subseteq \ell_p \subseteq \ell_q \subseteq \ell_{\infty} \).

**Proof.** If the sequence \( x = \{x_1, x_2, \cdots \} \in \ell_1 \), then we first show that \( \|x\|_{\ell_1} \leq \|x\|_{\ell_0} \leq \|x\|_{\ell_p} \leq \|x\|_{\ell_0} < \infty \). For \( p > 1 \) and \( x \in \ell_1 \), it follows that, for \( \forall n \in \mathbb{N} \),
   \[
   \sum_{k=1}^{n} |x_k|^p \leq \left( \sum_{k=1}^{n} |x_k| \right)^p \leq \left( \sum_{k=1}^{\infty} |x_k| \right)^p = \|x\|_{\ell_1}^p
   \]
   As \( n \to \infty \), the above equations yield: \( \|x\|_{\ell_1}^p \leq \|x\|_{\ell_0}^p \), which implies \( \|x\|_{\ell_0} \leq \|x\|_{\ell_1} \).
   For \( q > p > 1 \) and \( x \in \ell_p \), it follows that, for \( \forall n \in \mathbb{N} \),
   \[
   \sum_{k=1}^{n} |x_k|^q \leq \left( \sum_{k=1}^{n} |x_k|^p \right)^{\frac{q}{p}} \leq \left( \sum_{k=1}^{\infty} |x_k|^p \right)^{\frac{q}{p}} = \|x\|_{\ell_p}^q
   \]
   As \( n \to \infty \), the above equations yield: \( \|x\|_{\ell_1}^q \leq \|x\|_{\ell_p}^q \), which implies \( \|x\|_{\ell_p} \leq \|x\|_{\ell_0} \). Then, we consider a sequence \( \{x_k\} \) that belongs to \( \ell_q \) but not to \( \ell_p \) for \( 1 < p < q < \infty \). This establishes the strict inclusion as needed in the proof of the theorem.
   Next, let \( x \in \ell_q \) for \( q \in [1, \infty) \); then, \( |x_k| \leq \left( \sum_{k=1}^{\infty} |x_k|^q \right)^{\frac{1}{q}} = \|x\|_{\ell_q} \). Taking supremum over all \( k \in \mathbb{N} \), it follows that \( \|x\|_{\ell_{\infty}} \leq \|x\|_{\ell_q} \). We note that the discrete step function (i.e., the sequence \( \{1, 1, \cdots \} \)) belongs to \( \ell_{\infty} \) but not to \( \ell_q \) for \( q < \infty \). This establishes the strict inclusion as needed in the proof of the theorem. \qed

**Theorem 2.4.** *(Inclusion Theorem 2)* Let \((\mathbb{R}, \mathcal{B}, \nu)\) be a measure space, where the measure \( \nu \) is finite. Then, given \( 1 < p < q < \infty \), the following condition holds:
   \( L_1(\nu) \supseteq L_p(\nu) \supseteq L_q(\nu) \supseteq L_{\infty}(\nu) \).
Proof. The proof follows by an application of the integral form of Theorem 2.1 (Hölder Inequality).

**Theorem 2.5. (Inclusion Theorem 3)** Let $(\mathbb{R}, \mathcal{B}, \mu)$ be a measure space and the measure $\mu$ is either finite or $\sigma$-finite. Then, given $p \in [1, \infty]$, the following condition holds: $L_1(\mu) \cap L_\infty(\mu)$ is a subspace of $L_p(\mu)$.

Proof. Let $x \in L_1(\mu) \cap L_\infty(\mu)$. If $p = 1$ or if $p = \infty$, the proof is obvious that $x \in L_p(\mu)$. Therefore, let $p \in (1, \infty)$. Let $S \subseteq \mathbb{R}$ such that $|x(t)| \geq 1$ $\forall t \in S$. Since $x \in L_1(\mu)$, i.e., $\|x\|_{L_1} \triangleq \int_\mathbb{R} d\mu(t) \ |x(t)| < \infty$, it follows that the measure $\mu(S) < \infty$. Furthermore, since $x \in L_\infty(\mu)$, i.e., $\|x(t)\|_{L_\infty} \triangleq \text{ess sup}_{t \in \mathbb{R}} |x(t)| < \infty$, it follows that $|x(t)| \leq \|x(t)\|_{L_\infty} < \infty \mu$-a.e. on $S$. Therefore, for any $p \in (1, \infty)$, it follows that

$$
\left(\|x\|_{L_p}\right)^p = \int_\mathbb{R} d\mu(t) \ |x(t)|^p = \int_S d\mu(t) \ |x(t)|^p + \int_{\mathbb{R} \setminus S} d\mu(t) \ |x(t)|^p \\
\leq \int_S d\mu(t) \ |x(t)|^p + \int_{\mathbb{R} \setminus S} d\mu(t) \ |x(t)|^p \text{ because } |x(t)| < 1 \text{ on } \mathbb{R} \setminus S \\
\leq \int_S d\mu(t) \ \|x\|^p_{L_\infty} + \int_{\mathbb{R} \setminus S} d\mu(t) \ |x(t)| \\
= \mu(S) \ \|x\|^p_{L_\infty} \leq \|x\|_{L_1} < \infty
$$

The above inequality implies that $L_1(\mu) \cap L_\infty(\mu) \subseteq L_p(\mu)$. Since both $L_1(\mu)$ and $L_\infty(\mu)$ are vector spaces, their intersection $L_1(\mu) \cap L_\infty(\mu)$ is also a vector space. Hence, $L_1(\mu) \cap L_\infty(\mu)$ is a subspace of the vector space $L_p(\mu)$ for all $p \in [1, \infty]$.

\[ \text{Figure 2: Comparison of profiles of unit disk under different norms in } \mathbb{R}^2 \]

\{(Courty: Fig. 16 in Kreyszig p. 65)\)

**Example 2.1.** Let us consider the space $\mathbb{R}^2$ and the closed & bounded sets $S_p^1 \triangleq \{ x \in \mathbb{R}^2 : \|x\|_p = 1 \}$ for $p \in [1, \infty]$, where $\|x\|_p \triangleq (|x_1|^p + |x_2|^p)^{\frac{1}{p}}$ for $p \in [1, \infty)$ and $\|x\|_\infty \triangleq \max\{|x_1|, |x_2|\}$. Given the vector space $V = \mathbb{R}^2$, let us construct the
regions $\|x\|_{\ell_p}$ for $p \in [1, \infty]$. Referring to Figure 2, we notice that

\[
\{ x \in \mathbb{R}^2 : \|x\|_{\ell_1} \leq 1 \} \quad \text{(a rhombus)}
\]

\[
\subsetneq \{ x \in \mathbb{R}^2 : \|x\|_{\ell_2} \leq 1 \} \quad \text{(a circle)}
\]

\[
\subsetneq \{ x \in \mathbb{R}^2 : \|x\|_{\ell_4} \leq 1 \} \quad \text{(a smooth distorted circle)}
\]

\[
\subsetneq \{ x \in \mathbb{R}^2 : \|x\|_{\ell_\infty} \leq 1 \} \quad \text{(a square)}
\]

3 Completion of Normed Spaces

The notion of Hamel basis in finite-dimensional vector spaces has been introduced in Chapter 2. It is a purely algebraic concept. The concept of convergence of a series can be used to define a “basis” in the following definition.

**Definition 3.1.** (Schauder Basis) Let $(V, \| \cdot \|)$ be a normed space and let \( \{ e^k \} \) be a sequence in $V$ such that, for every $x \in V$, there exists a sequence of scalars \( \{ \alpha_k \} \) with the property

\[
\left\| \left( x - \sum_{k=1}^{n} \alpha_k e^k \right) \right\| \to 0 \quad \text{as } n \to \infty.
\]

Then, the sequence \( \{ e^k \} \) is called a Schauder basis for $(V, \| \cdot \|)$.

**Definition 3.2.** (Banach Space) A complete normed vector space (i.e., where every Cauchy sequence converges in the space) is called a Banach space.

**Lemma 3.1.** (Boundedness of a Cauchy Sequence) Every Cauchy sequence is bounded in a normed space.

**Proof.** Let $(V, \| \cdot \|)$ be a normed space. Let \( \{ x^k \} \) be a Cauchy sequence in $(V, \| \cdot \|)$. Then, for any given $\varepsilon > 0$, there exists $m(\varepsilon) \in \mathbb{N}$ such that $\|x^\ell - x^n\| \leq \varepsilon$ for all $\ell, n \geq m$. Since $\|x^m\| < \infty$, it follows by the triangular inequality and the fact that

\[
\forall n > m, \quad \|x^n\| = \|x^m + (x^n - x^m)\| \leq \|x^m\| + \|x^n - x^m\| \leq \|x^m\| + \varepsilon < \infty
\]

\[\blacklozenge\]

**Theorem 3.1.** (Completion of Finite-dimensional Normed Spaces) Let $(V, \| \cdot \|)$ be an $n$-dimensional normed vector space over a (complete) field $\mathbb{F}$, where $n \in \mathbb{N}$. Then, $(V, \| \cdot \|)$ is a Banach space.

**Proof.** We make use of the fact that any $n$-dimensional space $V$ over the field $\mathbb{F}$ is isomorphic to $\mathbb{F}^n$ for any $n \in \mathbb{N}$. Then, we use the method of induction to prove the theorem based on the convergence of an arbitrary Cauchy sequence in $\mathbb{F}^n$ to a point in $\mathbb{F}^n$. It follows that if $n = 1$, then $\mathbb{F}$ is complete in the sense of the usual metric, because of the hypothesis of completeness of the field $\mathbb{F}$. Next, we show that, for all $n \in \mathbb{N} \setminus \{1\}$, if the space $\mathbb{F}^{n-1}$ is complete in the sense of the usual metric, then the space $\mathbb{F}^n$ is also complete in the sense of the usual metric. Let \( \{ e^1, \ldots, e^n \} \) be a basis for the vector space $\mathbb{F}^n$ and let

\[
\delta_k \triangleq \inf_{\alpha_j \in \mathbb{F}^n} \|e^k - \sum_{j \neq k} \alpha_j e^j\| > 0 \quad \forall k \in \{1, \ldots, n\},
\]
which is the distance from the vector \( e^k \) to an \((n-1)\)-dimensional subspace of \( \mathbb{F}^n \) generated by the the remaining basis vectors, \( e^1, \ldots, e^{k-1}, e^{k+1}, \ldots, e^n \). Since the basis \( e^1, \ldots, e^n \) forms a linearly independent set, \( \delta_k > 0 \). Then,

\[
\delta \triangleq \min_k \delta_k > 0 \text{ because } \delta_k > 0 \ \forall k \in \{1, \ldots, n\}
\]

Let \( \{x^k\} \) be a Cauchy sequence in \( \mathbb{F}^n \), where each vector can be uniquely represented in terms of the basis \( \{e^1, \ldots, e^n\} \) as

\[
x^k = \sum_{j=1}^{k} \lambda^j_k e^j \quad \text{where } \lambda^j_k \in \mathbb{F}
\]

Then, for any \( \ell, m \in \mathbb{N} \), it follows that

\[
\|x^\ell - x^m\| = \left\| \sum_{j=1}^{n} (\lambda^j_\ell - \lambda^j_m) e^j \right\| \geq |\lambda^j_\ell - \lambda^j_m| \delta \quad \forall k \in \{1, \ldots, n\}
\]

Given that \( \lim_{\ell, m \to \infty} \|x^\ell - x^m\| = 0 \) for the Cauchy sequence \( \{x^k\} \) and the scalar \( \delta > 0 \), it follows that \( \lim_{\ell, m \to \infty} |\lambda^j_\ell - \lambda^j_m| = 0 \), i.e., the sequence \( \{\lambda^j_k\} \) converges to a scalar \( \lambda_k \) \( \forall k \in \{1, \ldots, n\} \).

Let us construct a vector \( x \triangleq \sum_{j=1}^{n} \lambda_j e^j \in \mathbb{F}^n \) and then show that \( x^\ell \to x \).

For all \( \ell \in \mathbb{N} \), it follows by the triangular inequality and homogeneity properties of a norm that

\[
\|x^\ell - x\| = \left\| \sum_{j=1}^{n} (\lambda^j_\ell - \lambda_j) e^j \right\| \leq \sum_{j=1}^{n} |\lambda^j_\ell - \lambda_j| \|e^j\|
\]

Since \( \lim_{\ell \to \infty} |\lambda^j_\ell - \lambda_j| = 0 \ \forall j \in \{1, \ldots, n\} \), it follows that \( \lim_{\ell \to \infty} \|x^\ell - x\| = 0 \).

Hence, the Cauchy sequence \( \{x^k\} \) converges to \( x \in \mathbb{F}^n \). \(\square\)

We present an alternative proof of Theorem 3.1.

**Proof.** Let \( \{y^k\} \) be an arbitrary Cauchy sequence of vectors in \( V \). We will show that \( \{y^k\} \) converges in \( V \), i.e., there exists \( y \in V \) such that \( \lim_{\ell \to \infty} \|x^\ell - x\| = 0 \); in other words, the limit of the sequence \( \{y^k\} \) is \( y \in V \).

Since \( \dim V = n \), we select a basis for \( V \) as \( \{v^1, \ldots, v^n\} \). Then each \( y^k \) is expressed in the form: \( y^k = \sum_{j=1}^{n} \alpha^k_j v^j \), where \( \alpha^k_j \in \mathbb{F} \ \forall j, k \). Since \( \{y^k\} \) is a Cauchy sequence of vectors in \( V \), it follows that \( \forall \varepsilon > 0 \exists n \in \mathbb{N} \) such that \( \|y_j - y_\ell\| < \varepsilon \) for all \( j, \ell > n \). Then, using Lemma 1.2, we assert that there exists a real constant \( c \in (0, \infty) \) such that

\[
\varepsilon > \|y^k - y^\ell\| = \left\| \sum_{j=1}^{n} (\alpha^k_j - \alpha^\ell_j) v^j \right\| \geq c \sum_{j=1}^{n} |(\alpha^k_j - \alpha^\ell_j)| \quad \text{(where } k, \ell > N\text{)}
\]

Division by \( c \in (0, \infty) \) yields

\[
|\alpha^k_m - \alpha^\ell_m| \leq \sum_{j=1}^{n} |(\alpha^k_j - \alpha^\ell_j)| < \frac{\varepsilon}{c} \quad \forall m = 1, \ldots, n \quad \forall k, \ell > N
\]

Therefore, each of the \( n \) sequences \( \alpha^k_m, \ m = 1, \ldots, n \) is Cauchy in \( \mathbb{F} \). Since the field \( \mathbb{F} \) is complete, each of these sequences of scalars converges; let us denote
the respective limits as $\alpha_m$, $m = 1, \cdots, n$. Using these $n$ limits, we construct $y = \sum_{m=1}^{n} \alpha_m v^m$. Therefore, $y \in V$ and

$$\|y^k - y\| = \left\| \sum_{j=1}^{n} (\alpha_j - \alpha_j) v^j \right\| \leq \sum_{j=1}^{n} |(\alpha_j - \alpha_j)| \|v^j\|$$

As $k \to \infty$, on the right side of the above equation, $\alpha_j \to \alpha_j$; hence, $\|y^k - y\| \to 0$, i.e., $y^k \to y$. This shows that $\{y^k\}$ is convergent in $V$. Since $\{y^k\}$ was originally assumed to be an arbitrary Cauchy sequence in $V$. The proof is now complete. \[\square\]

**Theorem 3.2.** The $\ell_p$-space over a complete field $\mathbb{F}$ (i.e., $\mathbb{R}$ or $\mathbb{C}$) is complete for $p \in [1, \infty]$.

**Proof.** First let us consider $p \in [1, \infty)$. Given a Cauchy sequence $\{x^k\}$ in $\ell_p$, we will show that $\{x^k\}$ converges to a sequence $x \in \ell_p$. Then, if $x^k = \{x^k_1, x^k_2, \cdots\}$, then $\|x^k - x^\ell\|_{\ell_p} \triangleq \left(\sum_{j=1}^{\infty} |\xi_j^k - \xi_j^\ell|^p\right)^{\frac{1}{p}}$ and it follows that

$$\lim_{k,\ell \to \infty} |\xi_j^k - \xi_j^\ell| \leq \lim_{k,\ell \to \infty} \left(\sum_{j=1}^{\infty} |\xi_j^k - \xi_j^\ell|^p\right)^{\frac{1}{p}} = 0 \quad \forall m \in \mathbb{N}$$

Hence, the sequence $\{\xi_m^k\}$ is Cauchy in $\mathbb{F}$ for each $m \in \mathbb{N}$ and therefore converges to a unique $\xi_m \in \mathbb{F}$. Now we show that the constructed sequence $x \triangleq \{\xi_1, \xi_2, \cdots\} \in \ell_p$.

It follows from Lemma 3.1 that $\exists M \in (0, \infty)$ which bounds the sequence $\{x^k\}$, i.e., $\|x^k\|_{\ell_p} \leq M \quad \forall k \in \mathbb{N}$. Hence,

$$\sum_{j=1}^{n} |\xi_j^k|^p \leq \|x^k\|_{\ell_p}^p \leq M^p \quad \forall n \in \mathbb{N}$$

Letting $k \to \infty$ in the finite sum in the left hand term of the above equation, the inequality $\sum_{j=1}^{\infty} |\xi_j|^p \leq M^p$ holds uniformly for all $n \in \mathbb{N}$. Then, letting $n \to \infty$, it follows that $\sum_{j=1}^{\infty} |\xi_j|^p \leq M^p < \infty$, i.e., $x \in \ell_p$.

Next we show that $x^k \to x$ as $k \to \infty$. Given $\varepsilon > 0$, $\exists \ell \in \mathbb{N}$ such that $\sum_{j=1}^{\ell} |\xi_j^k - \xi_j|^p \leq \left(\|x^k - x^\ell\|_{\ell_p}\right)^p \leq \varepsilon^p \quad \forall n, m > \ell$ and $\forall k \in \mathbb{N}$. Letting $m \to \infty$ in the finite sum in the left hand term of the inequality, it follows that $\sum_{j=1}^{\ell} |\xi_j^k - \xi_j|^p \leq \varepsilon^p \quad \forall n > \ell$ and $\forall k \in \mathbb{N}$. Now letting $k \to \infty$, it follows that $\|x^k - x\|_{\ell_p} < \varepsilon \quad \forall n > \ell$, which implies that $x^k \to x$ in $\ell_p$. This part completes the proof for $p \in [1, \infty)$.

It remains to show completion of $\ell_p$ for $p = \infty$, i.e., if $\{x^k\}$ is a Cauchy sequence in $\ell_\infty$, then $x^k \to x$ in $\ell_\infty$. Let $\{x^k\}$ be a Cauchy sequence in $\ell_\infty$. Then, $\lim_{k,\ell \to \infty} \|x^k - x^\ell\|_{\ell_\infty} = 0$. Hence, $\forall j \in \mathbb{N}$ $\exists \xi_j \in \mathbb{F}$ such that $\xi_j^k \to \xi_j$ uniformly in $j$ as $k \to \infty$. Now we show that the constructed sequence $x \triangleq \{\xi_1, \xi_2, \cdots\} \in \ell_\infty$.

Since $\{x^k\}$ is a Cauchy sequence, it follows from Lemma 3.1 that $\exists M \in (0, \infty)$ which bounds the sequence $\{x^k\}$, i.e., $\|x^k\|_{\ell_\infty} \leq M \quad \forall k \in \mathbb{N}$. Hence, $\|x^k\|_{\ell_\infty} \leq M$, from which it follows that $\|x\|_{\ell_\infty} \leq M$, implying that $x \in \ell_\infty$. The convergence of $x^k$ to $x$ follows directly from the fact that $\xi_j^k \to \xi_j$ uniformly in $j$ as $k \to \infty$. \[\square\]

**Definition 3.3.** (Space of Convergent Sequences) A subspace of $\ell_\infty$, where all sequences are convergent, is called the $c$-space. A subspace of $c$ is called $c_0$ if all sequences converge to zero.
Remark 3.1. By Lemma 3.1, it follows that all sequences in the c-space are bounded. Therefore, c is a subspace of \( \ell_\infty \).

Theorem 3.3. (Completion of the c Space) The space c of all convergent sequences over a complete field \( F \) is complete.

Proof. Since c is a subspace of \( \ell_\infty \) that is a complete space, it suffices to establish completeness of the space c by showing that c is closed in \( \ell_\infty \). Let \( \bar{c} \) be the closure of c in \( \ell_\infty \). Then, there exists a Cauchy sequence \( \{ x^k : k \in \mathbb{N} \} \) of sequences in c, where \( x^k = \{ \xi_1^k, \xi_2^k, \ldots \} \) and \( x^k \to x \) in the \( \ell_\infty \)-induced metric, i.e., \( x \in \bar{c} \), where \( x = \{ \xi_1, \xi_2, \ldots \} \). We will show that \( x \in c \).

For any \( \varepsilon > 0 \), \( \exists n_1 \in \mathbb{N} \) such that \( \| x^k - x \|_{\ell_\infty} < \frac{\varepsilon}{3} \) \( \forall k \geq n_1 \), which implies that \( |\xi_j^k - \xi_j| < \frac{\varepsilon}{3} \) \( \forall j \). Since \( x^k \equiv \{ \xi_1^k, \xi_2^k, \ldots \} \) is convergent in the space c, it follows that, \( \forall k \in \mathbb{N} \) and any \( \varepsilon > 0 \), \( \exists n_2(\varepsilon) \in \mathbb{N} \) such that \( |\xi_i^k - \xi_j^k| < \frac{\varepsilon}{3} \) \( \forall i, j \geq n_2 \). Then, it follows by triangular inequality that \( \forall i, j, k \geq \max(n_1, n_2) \)

\[
|\xi_i - \xi_j| \leq |\xi_i - \xi_i^k| + |\xi_i^k - \xi_j^k| + |\xi_j^k - \xi_j| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon
\]

which implies that the sequence \( x = \{ \xi_1, \xi_2, \ldots \} \) is convergent in c \( \Rightarrow x \in c \). Since the choice of x is arbitrary, it follows that \( c = \bar{c} \) and hence c is closed in \( \ell_\infty \). \( \square \)

Corollary 3.1. (Completion of the c0 Space) The space c0 of all sequences converging to 0 over a complete field \( F \) is complete.

Proof. The proof follows from that of Theorem 3.3. \( \square \)

Definition 3.4. (Space of Continuous Functions) Having the domain \([a, b] \subset \mathbb{R}\), the space of all continuous functions that belong to the \( L_p[a, b] \) space, \( p \in [1, \infty] \), is called the \( C_p[a, b] \)-space. Note that \( C_p[a, b] \subset L_p[a, b] \).

Theorem 3.4. (Completion of the \( C_\infty[a, b] \) Space) Let \([a, b] \subset \mathbb{R}\). The space \( C_\infty[a, b] \) of all continuous functions which belong to \( L_\infty[a, b] \) is complete, where the metric on \( C_\infty[a, b] \) is defined as:

\[
d_\infty(x, y) = \max_{t \in [a, b]} |x(t) - y(t)|
\]

Proof. Let \( \{ x^k \} \) be a Cauchy sequence in \( C_\infty[0, 1] \) \( \Rightarrow \left( \forall \varepsilon > 0 \ \exists n(\varepsilon) \in \mathbb{N} \right) \) such that \( d_\infty(x^k, x^\ell) < \varepsilon \) \( \forall k, \ell \geq n(\varepsilon) \). Then, \( \forall t_0 \in [0, 1] \), the sequence \( x^k(t_0) \) of scalars converges to a scalar that we call \( x(t_0) \). In this way, we construct a scalar-valued function \( x \) whose domain is the compact set \([0, 1] \). Since \( x^k \to x \) uniformly on \([0, 1] \) and since \( x^k \)'s are continuous on \([0, 1] \) and the convergence is uniform, it follows that the limit function \( x \) is continuous on \([0, 1] \). Therefore, \( x \in C_\infty[0, 1] \). \( \square \)

Example 3.1. The space \( C_p[0, 1] \) is not complete for any \( p \in [1, \infty] \). To see this, let us consider a sequence of continuous functions \( \{ 1, t, t^2, \ldots \} \) whose domain is \([0, 1] \), i.e., \( x^k(t) = t^k \) \( \forall t \in [0, 1] \); it is verified from the fact

\[
\lim_{k, t \to \infty} \int_0^1 dt \ |t^k - t^\ell|^p = 0
\]

that \( \{ 1, t, t^2, \ldots \} \) is a Cauchy sequence in the \( C_p[0, 1] \) space for any \( p \in [1, \infty] \). It is also seen that \( x^k \to x \) pointwise on \([0, 1] \) in the metric induced by the \( L_p \)-norm for \( p \in [1, \infty] \). The limit function \( x \) is obtained as:
Clearly, the limit function $x$ is not continuous on $[0, 1]$ and hence $x \notin C_p[0, 1]$ but $x \in L_p[0, 1]$. That is, the Cauchy sequence $\{1, t, t^2, \cdots\}$ does not converge in $C_p[0, 1]$ but it does converge to a discontinuous function in $L_p[0, 1]$.

**Remark 3.2.** The above example reveals the fact all norms are not equivalent in an infinite-dimensional normed space. This result is different from the result derived in Theorem 1.5 for finite-dimensional vector spaces.

### 4 Density and Separability of Normed Spaces

**Definition 4.1.** (Density) Let $(V, \| \cdot \|)$ be a normed vector space. A set $S \subseteq V$ is called dense in $V$ if any one (or both) of the following two conditions are satisfied for all $x \in V$: (i) $x \in S$, or (ii) $x$ is a limit point of $S$. Equivalently, $\forall x \in V$ and $\forall \varepsilon > 0 \exists y \in S$ such that $\|x - y\| < \varepsilon$.

**Remark 4.1.** For a dense set $S$ in $V$, the closure of $S$ in $V$, denoted as $\overline{S}$, is the space $V$ itself, i.e., $\overline{S} = V$.

**Definition 4.2.** (Separability) Let $(V, \| \cdot \|)$ be a normed vector space. Then, $V$ is called separable if there exists a countable and dense subset $S \subseteq V$, i.e., $S$ is countable and $\overline{S} = V$.

**Example 4.1.** The space $\mathbb{R}^1$ of the real line is separable because $\mathbb{Q} \subset \mathbb{R}$ is countable and $\mathbb{Q} = \mathbb{R}$.

**Proposition 4.1.** The $\ell_p$ space is separable for $p \in [1, \infty)$.

**Proof.** Without loss of generality, we assume that $\mathbb{R}$ (which can be naturally extended to the complex field $\mathbb{C}$) is the field of the vector space $\ell_p$. Let $S \subseteq \ell_p$ be the set of all sequences with finitely many non-zero rational elements and the rest being zeros. Hence $S$ is a countable set because the countable union of countable sets forms a countable set (see Chapter One).

To show that $S$ is dense in $\ell_p$, let $x \triangleq \{\xi_1, \xi_2, \cdots\} \in \ell_p$, for a given $p \in [1, \infty)$, i.e., $\sum_{k=1}^{\infty} |\xi_k|^p < \infty$. Then, it follows that, for any fixed $\varepsilon > 0 \exists n(\varepsilon) \in \mathbb{N}$ such that $\sum_{k=n+1}^{\infty} |\xi_k|^p < \frac{\varepsilon^p}{2}$. It is also seen that, for $\forall k \in \{1, 2, \cdots, n\}$, if $q_k \in \mathbb{Q}$ such that $|\xi_k - q_k|^p < \frac{\varepsilon^p}{2n}$ and if $y \triangleq \{q_1, q_2, \cdots, q_n, 0, 0, \cdots\}$, then $y \in S$ and it follows that

$$
\|x - y\|_{\ell_p}^p = \sum_{j=1}^{n} |\xi_j - q_j|^p + \sum_{j=n+1}^{\infty} |\xi_j|^p < \left( n \frac{\varepsilon^p}{2n} + \frac{\varepsilon^p}{2} \right) = \varepsilon^p \Rightarrow \|x - y\|_{\ell_p} < \varepsilon
$$

Therefore, the countable $S$ is dense in $\ell_p$ and hence $\ell_p$ is separable for $p \in [1, \infty)$.

**Proposition 4.2.** The $\ell_\infty$ space is not separable.

**Proof.** We need the following lemma to prove the proposition.

**Lemma 4.1.** Let $S$ be the set of all (infinite) sequences whose elements belong to the binary alphabet $\{0, 1\}$. Then, $S$ is an uncountable set.
Proof. We prove the lemma by contradiction. Let us assume that $S$ is a countable set. Then, $S$ must consist of countably many sequences, i.e., $S \triangleq \{s^1, s^2, \cdots \}$ and the $i^{th}$ sequence is $s^i = \{s^i_1, s^i_2, \cdots \}$, where $s^i_j \in \{0, 1\}$. Let us construct a sequence $\tilde{s} \triangleq \{\tilde{s}_1, \tilde{s}_2, \cdots \}$ such that the $i^{th}$ element $\tilde{s}_i$ of the sequence $\tilde{s}$ is 0 (or 1) if the $i^{th}$ element $s^i_i$ of the sequence $s^i \in S$ is 1 (or 0), respectively. So, $\tilde{s} \neq s^i \quad \forall i \in \mathbb{N}$. Hence, $\tilde{s} \notin S$. This is a contradiction of the original assumption that $S$ is countable. Therefore, $S$ is uncountable.

Now we proceed to prove the proposition by making use of Lemma 4.1. Let $x$ and $y$ be two distinct sequences in $S$, i.e., $x, y \in S$ such that $x \neq y$. Then, $\|x - y\|_{\ell_\infty} = 1$, i.e., the distance between any two distinct points in $S$ is 1 in the $\ell_\infty$ metric. Let $D \subsetneq \ell_\infty$ be nonempty and countable. Let $U$ be an open ball in $\ell_\infty$ with radius $\frac{1}{2}$ and its center at one of the points in $D$. Therefore, $U \cap S$ contains at most one point of $S$. Since $D$ is countable and $S$ is uncountable, the closure $\overline{D}$ cannot contain all points of $S$. Since $S$ is a subset of $\ell_\infty$, $\overline{D}$ is a strictly proper subset of $\ell_\infty$. That is, no countable subset of $\ell_\infty$ is dense in $\ell_\infty$. So, $\ell_\infty$ is not separable.

Remark 4.2. It follows from Theorems 4.1 and 4.2 that the $\ell_p$-space and the $\ell_\infty$-space cannot have the same topology because separability is a topological property (see Proposition 3.2 in Chapter 1).