ME 550. FOUNDATIONS OF ENGINEERING SYSTEMS ANALYSIS
Chapter #05: Fourier Theory and Analysis

Notation :

C – *The algebraic field of complex numbers*
N – *The set of positve integers*
R – *The algebraic field of real numbers*
Z – *The Abelian group of integers*

Definition 05-1: An orthonormal set $\{x^\alpha : \alpha \in \mathrm{I}\}$ in an inner product space V is called complete (or maximal) if: $\left(\left\langle x, x^\alpha \right\rangle = 0 \ \ \forall \alpha \in \mathrm{I}\right) \Rightarrow \ \ x = \underline{0}$ [Note: I is a non-empty index set which is at most countable, or uncountable).♦

Definition 05-2: A complete orthonormal set in a Hilbert space H is called an orthonormal basis of H. ♦

Definition 05-3: A linear mapping P of a vector space V into itself is called a projection if $P^2 = P$. ♦

Definition 05-4: A projection P on an inner product space V is called orthogonal if its range space and null space are orthogonal. ♦

Theorem 05-1: **(Fourier Series Theorem)**. Let $\{x^k : k \in \mathrm{Z}\}$ be a (countable) orthonormal set in a Hilbert space H. Then, the following statements are equivalent:

(i) $\{x^k : k \in \mathrm{Z}\}$ is an orthonormal basis of H.

(ii) (Fourier Series Expansion) $\forall x \in H,\ x = \sum_{k \in \mathrm{Z}} \left\langle x, x^k \right\rangle x^k$ (in the L_2-sense). [Note: The scalars $\left\langle x, x^k \right\rangle$ are called Fourier coefficients of x.]

(iii) (Parseval Equality) $\forall y, z \in H,\ \langle y, z\rangle = \sum_{k \in \mathrm{Z}} \left\langle y, x^k \right\rangle \left\langle x^k, z \right\rangle$ (in the usual sense).

(iv) $\forall x \in H,\ \|x\|^2 = \sum_{k \in \mathrm{Z}} \left|\left\langle x, x^k \right\rangle\right|^2$ (in the usual sense).

(v) If V is a subspace of H such that $\{x^k : k \in \mathrm{Z}\} \subseteq V$, then V is dense in H, i.e., closure$(V) = H$.

Proof of Theorem 05-1: We need the following lemmas to prove the theorem. (See Naylor and Sell pp. 307-312.)

Lemma 1 (The Bessel Inequality): Let $\{x^k : k \in \mathrm{Z}\}$ be an orthonormal set in an inner product space V. Then,

$$\forall x \in V,\ \sum_{k \in \mathrm{Z}} \left|\left\langle x, x^k \right\rangle\right|^2 \le \|x\|^2$$

Proof of Lemma 1: Consider the finite set $\{x^k : k = -n, -(n-1), \cdots -1, 0, 1, \cdots, (n-1), n\} \subset \{x^k : k \in \mathrm{Z}\}$. Then,

$$0 \le \left\| x - \sum_{k=-n}^{n} \left\langle x, x^k \right\rangle x^k \right\|^2 = \left\langle \left(x - \sum_{k=-n}^{n} \left\langle x, x^k \right\rangle x^k \right), \left(x - \sum_{\ell=-n}^{n} \left\langle x, x^\ell \right\rangle x^\ell \right) \right\rangle$$

$$= \langle x, x\rangle - \sum_{k=-n}^{n} \left\langle x, x^k \right\rangle \left\langle x^k, x \right\rangle - \sum_{\ell=-n}^{n} \left\langle x, x^\ell \right\rangle \left\langle x^\ell, x \right\rangle + \sum_{k=-n}^{n} \sum_{\ell=-n}^{n} \overline{\left\langle x, x^k \right\rangle} \left\langle x, x^\ell \right\rangle \left\langle x^k, x^\ell \right\rangle$$

$$= \|x\|^2 - \sum_{k=-n}^{n} \left| \left\langle x, x^k \right\rangle \right|^2 - \sum_{k=-n}^{n} \left| \left\langle x^k, x \right\rangle \right|^2 + \sum_{k=-n}^{n} \left| \left\langle x^k, x \right\rangle \right|^2 \text{ because } \left\langle x^k, x^\ell \right\rangle = \delta_{k\ell}$$ ♦

Lemma 2: Let $\{x^k : k \in Z\}$ be a countable orthonormal set in a Hilbert space H. Then,

(i) The infinite series $\sum_{k\in Z} \alpha_k x^k$ converges if and only if $\sum_{k\in Z} |\alpha_k|^2$ converges. [Note: $\alpha_k \in C$]

(ii) Let $\sum_{k\in Z} |\alpha_k|^2$ converge and let $x = \sum_{k\in Z} \alpha_k x^k = \sum_{k\in Z} \beta_k x^k$. Then, $\|x\|^2 = \sum_{k\in Z} |\alpha_k|^2$ and $\alpha_k = \beta_k \ \forall k \in Z$.

[Note: Completion of the space H is not necessary for part (ii)]

Proof of Lemma 2: **Part (i):** Assume $\sum_{k\in Z} \alpha_k x^k$ is convergent and let $x = \sum_{k\in Z} \alpha_k x^k$ implying that

$\lim_{\ell\to\infty} \left\| x - \sum_{k=-\ell}^{\ell} \alpha_k x^k \right\|^2 = 0$. Since the inner product is a continuous function, we have:

$$\langle x, x^k \rangle = \left\langle \sum_{\ell\in Z} \alpha_\ell x^\ell, x^k \right\rangle = \sum_{\ell\in Z} \alpha_\ell \langle x^\ell, x^k \rangle = \alpha_k .$$

Now, using Lemma 1 (Bessel Inequality), we have $\sum_{\ell\in Z} \left|\langle x, x^\ell \rangle\right|^2 = \sum_{\ell\in Z} |\alpha_\ell|^2 \le \|x\|^2 < \infty$.

Next let us assume convergence of $\sum_{k\in Z} |\alpha_k|^2$ and let $s^\ell \equiv \sum_{k=-\ell}^{\ell} \alpha_k x^k$. Then, it follows that $\left\|s^\ell - s^j\right\|^2 = \sum_{k=-\ell}^{-(j+1)} |\alpha_k|^2 + \sum_{k=j+1}^{\ell} |\alpha_k|^2$ implying that the sequence $\{s^\ell\}$ of partial sums is Cauchy convergent. Since the space H is complete, the sequence $\{s^\ell\}$ converges in H. [Note: We need H to be complete in this part].

Part (ii): We first prove that $\|x\|^2 = \sum_{k\in Z} |\alpha_k|^2$ in the following steps:

$$\|x\|^2 - \sum_{k=-\ell}^{\ell} |\alpha_k|^2 = \langle x, x \rangle - \left\langle x, \sum_{k=-\ell}^{\ell} \alpha_k x^k \right\rangle + \left\langle x, \sum_{k=-\ell}^{\ell} \alpha_k x^k \right\rangle - \left\langle \sum_{k=-\ell}^{\ell} \alpha_k x^k, \sum_{k=-\ell}^{\ell} \alpha_k x^k \right\rangle$$

$$= \left\langle x, \left(x - \sum_{k=-\ell}^{\ell} \alpha_k x^k \right) \right\rangle + \left\langle \left(x - \sum_{k=-\ell}^{\ell} \alpha_k x^k \right), \sum_{k=-\ell}^{\ell} \alpha_k x^k \right\rangle$$

$$\le \|x\| \left\| x - \sum_{k=-\ell}^{\ell} \alpha_k x^k \right\| + \left\| x - \sum_{k=-\ell}^{\ell} \alpha_k x^k \right\| \left\| \sum_{k=-\ell}^{\ell} \alpha_k x^k \right\| \quad \text{by Schwarz inequality}$$

$$= \left\| x - \sum_{k=-\ell}^{\ell} \alpha_k x^k \right\| \left(\|x\| + \left\| \sum_{k=-\ell}^{\ell} \alpha_k x^k \right\| \right) \le 2\|x\| \left\| x - \sum_{k=-\ell}^{\ell} \alpha_k x^k \right\| \to 0 \text{ as } \ell \to \infty .$$

Next let $x = \sum_{k\in Z} \alpha_k x^k = \sum_{k\in Z} \beta_k x^k$. Then, $0 = \lim_{\ell\to\infty} \left(\sum_{k=-\ell}^{\ell} \alpha_k x^k - \sum_{k=-\ell}^{\ell} \beta_k x^k \right)$ implies that

$$0 = \lim_{\ell\to\infty} \sum_{k=-\ell}^{\ell} (\alpha_k - \beta_k) x^k \Rightarrow \sum_{k\in Z} |\alpha_k - \beta_k|^2 = 0 \Rightarrow \alpha_k = \beta_k \ \forall k \in Z$$ ♦

Remark 05-1: Any rearrangement of a convergent series $\sum_{k\in Z} \alpha_k x^k$ is convergent and the limit is independent of the rearrangement. ♦

Lemma 3: Let $\Theta \equiv \left\{x^k : k = -n, -(n-1), \cdots -1, 0, 1, \cdots, (n-1), n\right\}$ be a finite orthonormal set in a Hilbert space H. Let V be the linear space spanned by Θ. Then the orthogonal projection of any $x \in H$ is given by: $Px \equiv \sum_{k=-n}^{n} \left\langle x, x^k \right\rangle x^k$

Proof of Lemma 3: We observe that P is a linear mapping of H into itself and $Px^\ell = \sum_{k=-n}^{n} \left\langle x^\ell, x^k \right\rangle x^k$ because of orthogonality, i.e., $\left\langle x^k, x^\ell \right\rangle = \delta_{k\ell}$. And $P^2 x = P\left(\sum_{k=-n}^{n} \left\langle x, x^k \right\rangle x^k \right) = \sum_{k=-n}^{n} \left\langle x, x^k \right\rangle P x^k = Px$. Hence, P is a projection.

Let us denote the range of P as $R(P)$ and $R(P) \subset V$ because Px is a linear combination of vectors in V. Conversely, let $x \in V$. Then, $x = \sum_{k=-n}^{n} \alpha_k x^k$ where $\alpha_k = \left\langle x, x^k \right\rangle$. Therefore, $Px = x \Rightarrow V \subset R(P)$. Hence, $R(P) = V$.

Next we show that P is an orthogonal projection. Let $y \in N(P)$, the null space of P and let $x = R(P)$, the range space of P. Then, $x \in Px$ and

$$\left\langle y, x \right\rangle = \left\langle y, Px \right\rangle = \left\langle y, \sum_{k=-n}^{n} \left\langle x, x^k \right\rangle x^k \right\rangle = \sum_{k=-n}^{n} \left\langle y, x^k \right\rangle \overline{\left\langle x, x^k \right\rangle}$$

$$= \sum_{k=-n}^{n} \left\langle \left\langle y, x^k \right\rangle x^k, x \right\rangle = \sum_{k=-n}^{n} \left\langle 0, x \right\rangle = 0$$ implies that the projection P is orthogonal. ♦

Corollary to Lemma 3: $\forall x \in H$ any choice of the scalars $\alpha_k : k = -n, -(n-1), \cdots -1, 0, 1, \cdots, (n-1), n$,

$$\left\| x - \sum_{k=-n}^{n} \left\langle x, x^k \right\rangle x^k \right\| \le \left\| x - \sum_{k=-n}^{n} \alpha_k x^k \right\|$$

Proof of Corollary to Lemma 3: By the Projection Theorem (proved in Chapter 04) we have:

$\|x - Px\| \le \|x - y\|$ $\forall x \in H$ and $\forall y \in V$ where such a vector $y \in V$ can be expressed as: $y = \sum_{k=-n}^{n} \alpha_k x^k$ ♦

Lemma 4: Let $\Theta \equiv \left\{x^k : k \in \mathbb{Z}\right\}$ be a countable orthonormal set in a Hilbert space H. Let V be the linear space closed subspace generated by Θ, i.e., V = closure of span $\left\{x^k : k \in \mathbb{Z}\right\}$. Then,

(i) $\forall x \in V$ can be uniquely expressed as: $x = \sum_{k \in \mathbb{Z}} \left\langle x, x^k \right\rangle x^k$

(ii) The mapping P defined by: $Py = \sum_{k \in \mathbb{Z}} \left\langle y, x^k \right\rangle x^k$ $\forall y \in H$ is the orthogonal projection of H onto V.

Proof of Lemma 4: Part (i): Let $x \in H$ be the limit of (finite) linear combinations of the vectors in Θ, i.e., $x = \lim_{\ell \to \infty} \sum_{k=-\ell}^{\ell} \alpha_k x^k$. By Corollary to Lemma 3, we have $\left\| x - \sum_{k=-\ell}^{\ell} \left\langle x, x^k \right\rangle x^k \right\| \le \left\| x - \sum_{k=-\ell}^{\ell} \alpha_k x^k \right\|$ $\forall \ell \in \mathbb{N}$. Since $\lim_{\ell \to \infty} \left\| x - \sum_{k=-\ell}^{\ell} \alpha_k x^k \right\| = 0$, we have $x = \sum_{k=-\ell}^{\ell} \left\langle x, x^k \right\rangle x^k$ and its uniqueness follows from Part (ii) of Lemma 2.

Part (ii): Let $y = Px$. It follows from Lemma 1 and Part (ii) of Lemma 2 that

$$\|Px\|^2 = \left\| \sum_{k \in \mathbb{Z}} \left\langle x, x^k \right\rangle x^k \right\|^2 = \sum_{k \in \mathbb{Z}} \left\| \left\langle x, x^k \right\rangle \right\|^2 \le \|x\|^2$$

Since P is linear, $\|Px-Py\| = \|P(x-y)\| \le \|x-y\|$ implies uniform continuity of P. Therefore, we can exchange the infinite summation and the projection operator P.

$$Px^{\ell} = \sum_{k=-\ell}^{\ell} \left\langle x^{\ell}, x^k \right\rangle x^k = x^{\ell} \quad \text{because } \left\langle x^{\ell}, x^k \right\rangle = \delta_{\ell k}\text{; and } P^2 x = \sum_{k\in Z} \left\langle x, x^k \right\rangle P x^k = \sum_{k\in Z} \left\langle x, x^k \right\rangle x^k = Px$$

$$\Rightarrow \left(P^2 - P\right)x = 0 \ \forall x \in H \ \Rightarrow P^2 - P = 0 \Rightarrow P^2 = P. \text{ Hence, } P \text{ is a projection.}$$

The proof of orthogonality of P is similar to that in Lemma 3. ♦

Now we proceed to prove Theorem 1 (The Fourier Series Theorem):

(i) $\Rightarrow$ **(ii)** Given a complete orthonormal set $\{x^k\}$ in the Hilbert space H, let V be the closed linear subspace of H generated by $\{x^k\}$. If $x \in V^{\perp}$, then $\left\langle x, x^k \right\rangle = 0 \ \forall k \in Z$. Since $\{x^k\}$ is a complete orthonormal set, it follows from Definition 05-1 that $x = 0 \Rightarrow V^{\perp} = \{0\} \Rightarrow V = H$, and hence, the orthogonal projection of V onto H is the identity operator I following the Projection Theorem in Chapter 04. $Ix = x = \sum_{k\in Z} \left\langle x, x^k \right\rangle x^k$.

(ii) $\Rightarrow$ **(iii)** Given $y = \sum_{k\in Z} \left\langle y, x^k \right\rangle x^k$ and $z = \sum_{k\in Z} \left\langle z, x^k \right\rangle x^k$ for any $y, z \in H$,

$$\langle y, z\rangle = \left\langle \sum_{k\in Z} \left\langle y, x^k \right\rangle x^k, \sum_{k\in Z} \left\langle z, x^k \right\rangle x^k \right\rangle = \sum_{k\in Z} \left\langle y, x^k \right\rangle \overline{\left\langle z, x^k \right\rangle} = \sum_{k\in Z} \left\langle y, x^k \right\rangle \left\langle x^k, z \right\rangle$$

(iii) $\Rightarrow$ **(iv)** $\|x\|^2 = \langle x, x\rangle = \sum_{k\in Z} \left\langle x, x^k \right\rangle \overline{\left\langle x, x^k \right\rangle} = \sum_{k\in Z} \left|\left\langle x, x^k \right\rangle\right|^2$

(iv) $\Rightarrow$ **(i)** If $\{x^k\}$ is not a complete orthonormal set $\{x^k\}$ in the Hilbert space H, then $\exists$ a unit vector $\tilde{x} \in H$ such that $\left\langle \tilde{x}, x^k \right\rangle = 0 \ \forall k \in Z$. It follows by (iv) that $\|\tilde{x}\|^2 = \sum_{k\in Z} \left|\left\langle \tilde{x}, x^k \right\rangle\right|^2 = 0$ which is a contradiction.

We have shown that the statements **(i)**, **(ii)**, **(iii)** and **(iv)** in Theorem 05-1 are equivalent. Next we show that the statements **(ii)** and **(v)** are equivalent that will complete the proof of Fourier Series Theorem.

(v) $\Rightarrow$ **(ii)** Given V dense in H, i.e., closure$(V) = H$, by Lemma 4, the projection of $x \in H$ onto closure(V) is:

$$x = \sum_{k\in Z} \left\langle x, x^k \right\rangle x^k$$

(ii) $\Rightarrow$ **(v)** Given $x = \sum_{k\in Z} \left\langle x, x^k \right\rangle x^k$, the projection of every $x \in H$ onto closure(V) is equal to x itself implying that closure$(V) = H$, i.e., V dense in H. ♦

Definition 05-5: Given $f \in L_1(\Re)$, i.e., $\int_{\Re} dt \left| f(t) \right| < \infty$, the Fourier transform of f is defined as:

$$\hat{f}(\xi) \equiv \int_{\Re} dt \ \exp(-i\, 2\pi\xi t)\ f(t) \quad \forall \xi \in \Re \qquad \text{Fourier Analysis Formula}$$

and if $\hat{f} \in L_1(\Re)$, i.e., $\int_{\Re} d\xi \left| \hat{f}(\xi) \right| < \infty$, the inverse Fourier transform of $\hat{f}$ is defined as:

$$f(t) \equiv \int_{\Re} d\xi \ \exp(i\, 2\pi\xi t)\ \hat{f}(\xi) \quad \forall t \in \Re \qquad \text{Fourier Synthesis Formula}$$ ♦

Theorem 05-2: (Plancherel Theorem). Let $f,g \in L_1(\Re)\bigcap L_2(\Re)$, i.e., $\int_{\Re} dt\left|f(t)\right| < \infty$ and $\int_{\Re} dt\left|f(t)\right|^2 < \infty$; and $\int_{\Re} dt\left|g(t)\right| < \infty$ and $\int_{\Re} dt\left|g(t)\right|^2 < \infty$. Then,

$$\left\langle f,g\right\rangle = \left\langle \hat{f},\hat{g}\right\rangle, \text{ i.e., } \int_{\Re} dt\, \bar{f}(t)g(t) = \int_{\Re} d\xi\, \bar{\hat{f}}(\xi)g(\xi). \text{ Then, it follows that}$$

$$\|f\|_{L_2} = \left\|\hat{f}\right\|_{L_2}, \text{ i.e., } \int_{\Re} dt\left|f(t)\right|^2 = \int_{\Re} d\xi\left|\hat{f}(\xi)\right|^2$$

Proof of Theorem 05-2: See Real Complex Analysis by Rudin (pp. 187-189).

Properties of Fourier Transform: The following properties hold for $\forall\, \alpha,\beta \in \mathrm{C}$ and $\forall\, f,g \in L_1(\Re)\bigcap L_2(\Re)$:

Linearity: $\alpha f(t) + \beta g(t) \;\leftrightarrow\; \alpha \hat{f}(\xi) + \beta \hat{g}(\xi)$

Symmetry: $f(t) \;\leftrightarrow\; \hat{f}(\xi) \;\Leftrightarrow\; \hat{f}(t) \;\leftrightarrow\; f(-\xi)$

Shifting: $f(t-t_o) \leftrightarrow \exp(-i2\pi\xi t_o)\hat{f}(\xi)$ and $\exp(i2\pi\xi_o t) f(t) \leftrightarrow \hat{f}(\xi-\xi_o)$ **Modulation Theorem**

Scaling: $f(\alpha t) \;\leftrightarrow\; \frac{1}{|\alpha|}\hat{f}(\frac{\xi}{\alpha}) \quad \forall \alpha \neq 0$

Differentiation: $\frac{\partial^n f(t)}{\partial t^n} \;\leftrightarrow\; (i2\pi\xi)^n\, \hat{f}(\xi)$ and $(-i2\pi t)^n f(t) \;\leftrightarrow\; \frac{\partial^n \hat{f}(\xi)}{\partial \xi^n}$ provided the transforms exist.

Integration: $\int_{-\infty}^{t} d\tau\, f(\tau) \;\leftrightarrow\; \frac{\hat{f}(\xi)}{i2\pi\xi}$ provided that $\hat{f}(0) = 0$.

Moments: $(-i2\pi)^n \int_{-\infty}^{\infty} d\tau\, \tau^n f(\tau) = \frac{\partial^n \hat{f}(\xi)}{\partial \xi^n} \quad \forall n \in \mathrm{N}\bigcup\{0\}$ provided that the integral and the derivative exist.

Convolution: $f(t) * g(t) \;\leftrightarrow\; \hat{f}(\xi)\,\hat{g}(\xi) \quad and \quad f(t)\,g(t) \;\leftrightarrow\; \hat{f}(\xi) * \hat{g}(\xi)$

FOURIER SERIES (CONTINUOUS-TIME SERIES EXPANSION)

The objective here is to express a periodic function $f(t)$ of period $T > 0$, i.e., $f(t+T) = f(t)\ \forall t \in \Re$, as a (countable) linear combination of complex exponentials:

$$f(t) = \sum_{k\in \mathrm{Z}} \exp(i\,2\pi kt/T)\, \hat{f}[k] \text{ with } \hat{f}[k] \equiv \frac{1}{T}\int_{-T/2}^{T/2} dt\, \exp(-i\ 2\pi kt/T)\, f(t)$$

provided that the coefficients $\hat{f}[k]$ are well defined. If $f \in L_2[-\frac{T}{2},\frac{T}{2})$ but not necessarily continuous everywhere on $[-T/2,T/2)$, the series $\sum_{k\in \mathrm{Z}} \exp(i\,2\pi kt/T)\,\hat{f}[k]$ converges to $f(t)$ in the L_2 sense, i.e., $\left\| f(t) - \sum_{k\in \mathrm{Z}} \exp(i\,2\pi kt/T)\,\hat{f}[k] \right\|_{L_2} = 0$. In other words, if we define $f_N(t) \equiv \sum_{k=-N}^{N} \exp(i\,2\pi kt/T)\,\hat{f}[k]$ for any $N \in \mathrm{N}$, then $\|f - f_N\|_{L_2} \equiv \int dt\left|f(t) - f_N(t)\right|^2 \to 0$ as $N \to \infty$. However, uniform convergence may not be guaranteed. To see it, note that $f_N(t)$ may overshoot or undershoot near the point of discontinuity and the amount depends on the number of terms N used in the approximation.

HW 05-1: Define $\psi[k,t] \equiv \frac{1}{\sqrt{T}}\exp(i\,2\pi k t/T)$ for $t \in [-T/2, T/2)$ and $k \in \mathrm{Z}$. Show that $\{\psi[k,t] : k \in \mathrm{Z}\}$ forms an orthonormal basis for the space $L_2\left(\frac{-T}{2}, \frac{T}{2}\right)$.

HW 05-2: Establish the Parseval relation $\langle f, g\rangle_{[-T/2, T/2)} = \langle \hat{f}, \hat{g}\rangle_{\mathrm{Z}}$ implying that $\|f\|_{L_2\left(\frac{-T}{2},\frac{T}{2}\right)} = \|\hat{f}\|_{\ell_2}$.

Remark 05-2: It is possible to derive the approximation property in view of the orthogonal projection as:

$$\left\| f(\bullet) - \sum_{k=-N}^{N} \langle \psi[k,\bullet], f(\bullet)\rangle \psi[k,\bullet] \right\|_{L_2\left[\frac{-T}{2},\frac{T}{2}\right)} \le \left\| f(\bullet) - \sum_{k=-N}^{N} \alpha_k\, \psi[k,\bullet] \right\|_{L_2\left[\frac{-T}{2},\frac{T}{2}\right)}$$

where $\{\alpha_k\}$ is an arbitrary sequence that can be optimized to yield the least error norm. ♦

Remark 05-3: In addition to decomposition of periodic signals, problems defined on compact support (i.e., of finite size) can be analyzed by Fourier series. However, the critical issue is the introduction of the discontinuity at the boundary since periodization of a continuous signal on an interval may often result in a discontinuous periodic signal. ♦

Definition 05-6: Let $H_T \equiv L_2[t_o, t_o + T)$, i.e., the set of complex-valued Lebesgue-measurable square-integrable T -periodic functions for given $t_o \in \Re$ and $T \in (0,\infty)$. Then, we define:

$$\left.\begin{aligned} \|f\|_T^2 &\equiv \int_{t_o}^{t_o+T} dt\, |f(t)|^2 < \infty \\ \langle f, g\rangle_T &\equiv \int_{t_o}^{t_o+T} dt\, \bar{f}(t) g(t) \end{aligned}\right\} \ \forall\, f, g \in H_T$$ ♦

Remark 05-4: $H_T \equiv L_2[t_o, t_o + T)$ is a separable Hilbert space implying that

(i) H_T is complete, i.e., every Cauchy sequence in H_T converges in H_T .

(ii) H_T has a countable orthonormal basis.

(iii) H_T is isometrically isomorphic to other Hilbert spaces of the same Hilbert dimension (e.g., the ℓ_2 space). ♦

Let us define $\psi_k(t) \equiv \exp(i\,2\pi k t/T)$ for $k \in \mathrm{Z}$ implying that $\|\psi_k\|_T^2 = T$ and $\langle \psi_k, \psi_\ell\rangle = T\delta_\ell^k$. Let $f_N \in H_T$ be a (finite) linear combination of ψ_k 's , i.e., for arbitrary $N \in \mathrm{N}$

$$f_N(t) \equiv \frac{1}{T}\sum_{k=-N}^{N} c_k \psi_k(t) \quad \Rightarrow \quad c_k = \langle \psi_k, f_N\rangle_T = \int_{t_o}^{t_o+T} dt \exp(-i\,2\pi k t/T)\, f_N(t)$$

$$\|f_N\|_T^2 = \frac{1}{T}\sum_{k=-N}^{N} |c_k|^2$$

Now, we are in a position to define a single "mother" function $\psi(t) \equiv \exp(i\,2\pi\, t/T)$ to generate an orthonormal base $\{\psi_k : k \in \mathrm{Z}\}$ for H_T where $\psi_k(t) \equiv \psi(kt)$. Given a sequence of vectors $\{f_k : k \in \mathrm{N}\}$ in H_T (which is a complete space) converges to a function $f \in H_T$ in the L_2 sense following the Fourier Series Theorem. Hence,

$$f(t) = \frac{1}{T}\sum_{k \in \mathrm{Z}} c_k \psi_k(t) \ \text{ with } \ c_k = \langle \psi_k, f\rangle_T = \int_{t_o}^{t_o+T} dt \exp(-i\,2\pi k t/T)\, f(t)$$

Because of the isometric isomorphism between H_T and $\ell_2(\mathrm{Z})$, a T -periodic signal can be represented by its Fourier coefficients. We can also write $\langle \psi_k, f\rangle = \psi_k^* f$ where ψ_k^* is the adjoint of ψ_k and this results in the following resolution of identity: $\frac{1}{T}\sum_{k\in Z}\psi_k\psi_k^* = I$ with the set $\left\{\frac{1}{\sqrt{T}}\psi_k : k\in Z\right\}$ of normalized functions forming an orthonormal basis for H_T.

Next let us choose $t_o = -T/2$ so that the interval integration becomes symmetric, and define the function

$$_T\hat{f}[k] = \int_{-T/2}^{T/2} dt \exp(-i\,2\pi kt/T)\,f(t) \quad \text{for any } k\in Z \text{ and } T\in(0,\infty)$$

The coefficients are determined as:

$$c_k = \int_{-T/2}^{T/2} dt \exp(-i\,2\pi\xi_k t)\,f(t) = \hat{f}(\xi_k) \quad \text{where } \xi_k \equiv \frac{k}{T}$$

Note that, as $T\to\infty$, $_T\hat{f}[k]$ converges to the Fourier integral transform of $f(t)$ defined as:

$$\hat{f}(\xi) \equiv \int_{\Re} dt \exp(-i\,2\pi\xi t)\,f(t)$$

Next, for a fixed $T\in(0,\infty)$, we have $\Delta\xi_k \equiv \xi_{k+1}-\xi_k = \frac{k+1}{T}-\frac{k}{T} = \frac{1}{T}$. Then.

$$f(t) = \frac{1}{T}\sum_{k\in Z} c_k\,\psi_k(t) = \sum_{k\in Z}\Delta\xi_k\,\psi_k(t)\exp(i\,2\pi\xi_k t)\,\hat{f}_T(\xi_k)$$

is a Riemann sum that, in the limit $T\to\infty$, becomes the inverse Fourier transform

$$f(t) \equiv \int_{\Re} d\xi \exp(i\,2\pi\xi t)\,\hat{f}(\xi)$$

Similarly, the energy relation $\|f\|^2_{L_2[-T/2,\,T/2]} = \frac{1}{T}\sum_{k\in Z}|c_k|^2 = \sum_{k\in Z}\Delta\xi_k\left|\hat{f}_T(\xi_k)\right|^2$ is also a Riemann sum that, in the limit $T\to\infty$, becomes $\|f\|^2_{L_2(\Re)} \equiv \int_{\Re} dt\,|f(t)|^2 = \int_{\Re} d\xi\,\left|\hat{f}(\xi)\right|^2 = \left\|\hat{f}\right\|^2_{L_2(\Re)}$ provided that $f\in L_1(\Re)\cap L_2(\Re)$. This relation is derived from the Plancherel theorem expressed as $\langle f,g\rangle_{L_2(\Re)} = \int_{\Re} dt\,\bar{f}(t)g(t) = \int_{\Re} d\xi\,\bar{\hat{f}}(\xi)\,\hat{g}(\xi) = \langle\hat{f},\hat{g}\rangle$. The implication is that the Fourier transform is an isometric isomorphism from $L_2(\Re)$ onto $L_2(\Re)$.

Definition 05-7 (Dirac Delta Distribution and Impulse Trains): Given any fixed $\varepsilon\in(0,\infty)$, define

$$\delta_\varepsilon(t) \equiv \begin{cases} 1/\varepsilon & \text{for } t\in[0,\varepsilon) \\ 0 & \text{otherwise} \end{cases}$$

Then, the Dirac delta distribution is defined as: $\delta(t) \equiv \lim_{\varepsilon\to 0^+}\delta_\varepsilon(t)$. We can also use any smooth function $\psi(t)$ with $\int_{\Re} dt\,\psi(t) = 1$ to define $\delta(t) \equiv \lim_{\varepsilon\to 0^+}\frac{\psi(t/\varepsilon)}{\varepsilon}$. In essence, the Dirac delta distribution $\delta(t)$ has the following properties:

- $\int_{\Re} d\tau\,\delta(\tau) = 1$
- $\begin{cases} \int_{\Re} d\tau\; f(t-\tau)\,\delta(\tau) = \int_{\Re} d\tau\; f(\tau)\,\delta(t-\tau) = f(t) \\ \Rightarrow f(t)*\delta(t-t_o) = f(t-t_o) \Rightarrow \begin{cases} \delta(t-t_o) & \leftrightarrow & \exp(-i\,2\pi\xi t_o) \\ \exp(i\,2\pi\xi_o t) & \leftrightarrow & \delta(\xi-\xi_o) \end{cases} \end{cases}$

The train of Dirac delta distributions spaced $T>0$ apart is defined as:

$$\sigma_T(t) \equiv \lim_{N\to\infty}\sum_{\ell=-N}^{N}\delta(t-\ell T) = \sum_{\ell\in Z}\delta(t-\ell T)$$ ♦

Theorem 05-3: **(Poisson Sum Formula)**. Let $f(t)$ be a rapidly decaying function such that the following integral of an infinite sum

$$\int_{\Re} d\tau\, \sigma_T(t-\tau)\, f(\tau) = \int_{\Re} d\tau \lim_{N\to\infty} \sum_{\ell=-N}^{N} \delta(t-\tau-\ell T)\, f(\tau)$$

converges uniformly to a T -periodic function: $f_T(t) \equiv \lim_{N\to\infty} \sum_{\ell=-N}^{N} f(t-\ell T)$.

Then, $f_T(t) = \frac{1}{T} \sum_{k\in Z} \hat{f}\left(\frac{k}{T}\right) \exp(i\,2\pi kt/T)$.

Proof of Theorem 05-3: $\hat{f}_T[k] = \frac{1}{T} \int_{-T/2}^{T/2} dt\, \exp(-i\,2\pi kt/T)\, f_T(t)$ and $f_T(t) = \sum_{k\in Z} \hat{f}_T[k] \exp(i\,2\pi kt/T)$

$$\Rightarrow \quad f_T(t) = \sum_{k\in Z} \frac{1}{T} \int_{-T/2}^{T/2} d\tau\, \exp(-i\,2\pi k\tau/T)\, f_T(\tau) \exp(i\,2\pi kt/T)$$

Now, $\int_{-T/2}^{T/2} d\tau\, \exp(-i\,2\pi k\tau/T)\, f_T(\tau) = \int_{-T/2}^{T/2} d\tau\, \exp(-i\,2\pi k\tau/T) \sum_{\ell\in Z} f(t-\ell T)$

$= \sum_{m\in Z} \int_{-(2m-1)T/2}^{(2m+1)T/2} d\tau\, \exp(-i\,2\pi k\tau/T)\, f(\tau)$ [Absolute convergence of the integrand summand allow exchange of integral and sum.]

$= \int_{\Re} d\tau\, \exp(-i\,2\pi k\tau/T)\, f(\tau) = \hat{f}\left(\frac{k}{T}\right)$

Therefore, $f_T(t) = \frac{1}{T} \sum_{k\in Z} \hat{f}\left(\frac{k}{T}\right) \exp(i\,2\pi kt/T)$ ♦

Corollary 1 to Theorem 05-3: $\sum_{\ell\in Z} f(\ell) = \sum_{k\in Z} \hat{f}(k)$

Proof of Corollary 1: Setting $t=0$ and $T=1$, we have $f_T(0) = \sum_{\ell\in Z} f(-\ell) = \sum_{\ell\in Z} f(\ell)$ by definition, and by Poisson Sum Formula, $f_T(0) = \sum_{k\in Z} \hat{f}(k)$. Proof follows from these two equalities. ♦

Corollary 2 to Theorem 05-3: $\hat{\sigma}_T(\xi) = \frac{1}{T} \sum_{k\in Z} \delta\left(\xi - \frac{k}{T}\right)$

Proof of Corollary 2: Given $\sigma_T(t) \equiv \sum_{\ell\in Z} \delta(t-\ell T)$, by substitution of $f(t)$ with $\delta(t)$ Poisson Sum Formula yields

$$\sigma_T(t) = \frac{1}{T} \sum_{k\in Z} \hat{\delta}\left(\frac{k}{T}\right) \exp(i\,2\pi kt/T) = \frac{1}{T} \sum_{k\in Z} \exp(i\,2\pi kt/T)$$

because $\hat{\delta}\left(\frac{k}{T}\right) = \int_{\Re} d\tau\, \delta(\tau) \exp(-i\,2\pi k\tau/T) = 1 \quad \forall k\in Z$. Proof follows by taking Fourier transform of both sides:

$\hat{\sigma}_T(\xi) = \frac{1}{T} \sum_{k\in Z} \delta\left(\xi - \frac{k}{T}\right)$ ♦

Remark 05-5: (Sampling of Continuous Signals): Let $f_T(t)$ be the sampled version of a continuous signal $f(t)$ defined as:

$$f_T(t) = f(t)\sigma_T(t) = \sum_{\ell\in Z} f(\ell T)\delta(t-\ell T)$$

Using the modulation theorem and Corollary 2 to Poisson Sum Formula, we have

$$\hat{f}_T(\xi) = \hat{f}(\xi) * \hat{\sigma}_T(\xi) = \hat{f}(\xi) * \frac{1}{T} \sum_{k\in Z} \delta(\xi - \tfrac{k}{T}) = \frac{1}{T} \sum_{k\in Z} \hat{f}(\xi - \tfrac{k}{T})$$

Hence, $\hat{f}_T(\xi)$ is $\frac{1}{T}$ -periodic. This can be shown in an alternative way by the Poisson Sum Formula. Define

$$g_\theta(t) \equiv f(t) \exp(-i2\pi\theta t) \quad \Rightarrow \quad \hat{g}_\theta(\xi) \equiv \hat{f}(\xi+\theta)$$

Setting $t=0$ in Poisson Sum Formula, we have

$$\sum_{\ell\in Z} g_\theta(\ell T) = \frac{1}{T} \sum_{k\in Z} \hat{g}_\theta\left(\frac{k}{T}\right)$$

Substituting θ by ξ and switching the sign of k yield

$$\sum_{\ell\in Z} f(\ell T)\exp(-i2\pi\xi\ell T) = \frac{1}{T}\sum_{k\in Z}\hat{f}\left(\xi-\frac{k}{T}\right)$$

The left hand side is $\hat{f}_T(t) = \int_{\Re} dt\left(\sum_{\ell\in Z} f(\ell T)\delta(t-\ell T)\right)\exp(-i2\pi\xi t) = \sum_{\ell\in Z} f(\ell T)\exp(-i2\pi\xi\ell T)$. ♦

Theorem 05-4: (The Shannon Sampling Theorem): Let a continuous signal $f \in L_2(\Re)$ be band-limited by Ω, i.e., $\hat{f}(\xi) = 0\ \forall \xi \notin [-\Omega,\Omega]$. Then, $f(t)$ can be perfectly reconstructed from its samples taken at instants $t_\ell \equiv \frac{\ell}{2\Omega} : \ell \in Z$, by the following interpolation formula:

$$f(t) = \sum_{\ell\in Z}\left(\frac{Sin(2\pi\Omega(t-t_\ell))}{2\pi\Omega(t-t_\ell)}\right) f(t_\ell)$$

Proof of Theorem 05-4: $\left\|\hat{f}\right\|^2_{L_2[-\Omega,\Omega]} = \int_{-\Omega}^{\Omega} d\xi \left|\hat{f}(\xi)\right|^2 = \int_{-\infty}^{\infty} d\xi \left|\hat{f}(\xi)\right|^2 = \left\|\hat{f}\right\|^2_{L_2(\Re)} = \|f\|^2_{L_2(\Re)} < \infty$.

Since $\hat{f} \in L_2[-\Omega,\Omega]$, $\hat{f}(\xi)$ can be expanded in a Fourier series in the interval $[-\Omega,\Omega]$ as:

$$\hat{f}(\xi) = \frac{1}{2\Omega}\sum_{\ell\in Z} c_\ell \exp(-i2\pi\xi t_\ell)$$

where the Fourier coefficients c_ℓ are identically equal to the samples of the continuous signal $f(t)$ taken at the instants $t_\ell \equiv \frac{\ell}{2\Omega} : \ell \in Z$, i.e., $c_\ell = \int_{-\Omega}^{\Omega} d\xi \exp(i2\pi\xi t_\ell)\hat{f}(\xi) = f(t_\ell)$. Hence, by equality in the L_2 sense,

$$f(t) = \int_{\Re} d\xi \exp(i2\pi\xi t)\hat{f}(\xi) = \int_{-\Omega}^{\Omega} d\xi \exp(i2\pi\xi t)\hat{f}(\xi) = \frac{1}{2\Omega}\int_{-\Omega}^{\Omega} d\xi \exp(i2\pi\xi t)\sum_{\ell\in Z} f(t_\ell)\exp(-i2\pi\xi t_\ell)$$

$$= \frac{1}{2\Omega}\sum_{\ell\in Z}\int_{-\Omega}^{\Omega} d\xi \exp(i2\pi\xi(t-t_\ell))\, f(t_\ell)$$ Note: Exchange of integral and sum holds due to uniform convergence

Therefore, $f(t) = \sum_{\ell\in Z}\frac{Sin(2\pi\Omega(t-t_\ell))}{2\pi\Omega(t-t_\ell)} f(t_\ell)$ where the equality holds in the L_2 sense. Since band-limited signals are analytic, it suffices to say that $f(\bullet)$ is a continuous function in the entire complex plane. Therefore, the above equality also holds pointwise. ♦

We now present the following corollary for an alternative interpretation of the Shannon Sampling Theorem as a series expansion of band-limited signals on an orthonormal basis.

Corollary 1 to Theorem 05-4: Let us define $\varphi_\ell(t) \equiv \sqrt{\frac{1}{T}} Sinc_T(t-\ell T)$ where $T = \frac{1}{2\Omega}$ and $Sinc_T(t) \equiv \frac{Sin(\pi t/T)}{\pi t/T}$.

Then,

(i) $\hat{\varphi}_\ell(\xi) = \begin{cases}\sqrt{T}\exp(-i2\pi\xi\ell T) & for\ \xi \in [-\Omega,\Omega] \\ 0 & otherwise\end{cases}$

(ii) $f(t) = \sum_{\ell\in Z}\langle f,\varphi_\ell\rangle\varphi_\ell(t)$ (equivalent to the interpolation formula of the Sampling Theorem)

Proof of Corollary 1 to Theorem 05-4:

(i) Fourier inverse $[\hat{\varphi}_\ell(\xi)]$ is $\sqrt{T}\int_{-\Omega}^{\Omega} d\xi \exp(i2\pi\xi t)\exp(-i2\pi\xi\ell T) = \sqrt{T}\int_{-\Omega}^{\Omega} d\xi \exp(i2\pi\xi(t-\ell T)) = \frac{1}{\sqrt{T}} Sinc_T(t-\ell T)$.

Therefore, Fourier transform of $\varphi_\ell(t)$ is $\hat{\varphi}_\ell(\xi)$ as defined above.

(ii) $\langle\hat{\varphi}_k,\hat{\varphi}_m\rangle = \int_{\Re} d\xi\, \overline{\hat{\varphi}_k}(\xi)\hat{\varphi}_m(\xi) = \frac{1}{2\Omega}\int_{-\Omega}^{\Omega} d\xi \exp(i2\pi\xi(\ell-m)T)) = \begin{cases}1 & \text{if } \ell = m \\ 0 & \text{if } \ell \neq m\end{cases}$

By Parseval identity, $\langle \varphi_\ell, \varphi_m \rangle = \langle \hat{\varphi}_\ell, \hat{\varphi}_m \rangle = \begin{cases} 1 & \text{if } \ell = m \\ 0 & \text{if } \ell \neq m \end{cases} \Rightarrow \{\varphi_\ell(t) : \ell \in \mathrm{Z}\}$ is an orthonormal basis of the space of $[-\Omega, \Omega]$-band-limited signals over, which is a closed subspace of $L_2(\Re)$. Hence, every $[-\Omega, \Omega]$-band-limited signal f can be expressed as: $f(t) = \sum_{\ell \in \mathrm{Z}} \langle f, \varphi_\ell \rangle \varphi_\ell(t)$ ♦

Remark 05-6: Referring to Remark 05-5, the sampled version $f_T(t)$ of the band-limited signal $f(t)$ has its Fourier transform: $\hat{f}_T(\xi) = \frac{1}{T} \sum_{k \in \mathrm{Z}} \hat{f}(\xi - \frac{k}{T})$. Since the sampling frequency is greater than or equal to 2Ω, $\hat{f}(\xi - k\Omega)$ and $\hat{f}(\xi - m\Omega)$ do not overlap for $k \neq m$. To recover $\hat{f}(\xi)$, it suffices to retain the term involving $k = 0$ in the summation term and normalize it by T. To this end, we construct an ideal low-pass filter with whose Fourier transform is $T = \frac{1}{2\Omega}$ for $\xi \in [-\Omega, \Omega]$ and zero elsewhere. Its time impulse response is:

$$Sinc_T(t) = T \int_{-\Omega}^{\Omega} d\xi \exp(-i2\pi\xi t) = \frac{T}{-i2\pi t}\left(\exp(-i2\pi\Omega t) - \exp(i2\pi\Omega t)\right) = \frac{Sin(\pi t / T)}{\pi t / T}$$

Convolving f_T with $Sinc_T$ filters out the repeated spectrums, i.e., terms with $k \neq 0$ in the summation $\sum_{k \in \mathrm{Z}} \hat{f}(\xi - \frac{k}{T})$. Since $f_T(t)$ is a sum of Dirac delta functions weighted by $f(\ell T)$, the convolution results in a weighted sum of shifted impulse responses:

$$\left(\sum_{\ell \in \mathrm{Z}} f(\ell T) \delta(t - \ell T) \right) * Sinc_T(t) = \sum_{\ell \in \mathrm{Z}} f(\ell T)\, Sinc_T(t - \ell T)$$

This shows that $f(t)$ can be recovered. ♦

DISCRETE-TIME FOURIER TRANSFORM (DISCRETE-TIME INTEGRAL TRANSFORM)

Definition 05-8: The discrete-time Fourier transform (DTFT) of a signal sequence $\{f[\ell]\} \in \ell_1(\mathrm{Z})$, i.e., $\sum_{\ell \in \mathrm{Z}} |f_\ell| < \infty$, is defined as:

$$\hat{f}^D(i2\pi\xi) = \sum_{\ell \in \mathrm{Z}} \exp(-i2\pi\xi\ell T) f[\ell] < \infty \text{ (which is } 2\pi\text{-periodic)}$$

The inverse discrete-time Fourier transform (IDTFT) of $\hat{f}^D(\bullet)$ is defined as:

$$f[\ell] \equiv \int_{-\pi}^{\pi} d\xi \exp(i2\pi\xi\ell) \hat{f}^D(\exp(i2\pi\xi)$$ ♦

Remark 05-7: Since $\ell_1(\mathrm{Z}) \subset \ell_2(\mathrm{Z})$, the signal sequence $\{f[\ell]\} \in \ell_2(\mathrm{Z})$. ♦

Remark 05-8: The above expressions for DTFT and IDTFT are duals of those for Fourier Series. If the sequence $\{f[\ell]\}$ is obtained by sampling a continuous signal $f(t)$ at instants ℓT, then $f[\ell] = f(\ell T)$ will have its DTFT related to the Fourier transform $\hat{f}(\xi)$ of $f(t)$. The Fourier transform of the sampled version is equal to:

$$\hat{f}_T(\xi) = \sum_{\ell \in \mathrm{Z}} \exp(-i2\pi\xi\ell T) f(\ell T) = \frac{1}{T} \sum_{k \in \mathrm{Z}} \hat{f}\left(\xi - \frac{k}{T}\right)$$

If $f[\ell] = f(\ell T)$, then $\hat{f}_T(\xi) = \hat{f}^D(i2\pi\xi)$ and it follows from the definition of DTFT that

$$\hat{f}^D(i2\pi\xi) = \frac{1}{T} \sum_{k \in \mathrm{Z}} \hat{f}\left(\xi - \frac{k}{T}\right)$$ ♦

DISCRETE-TIME FOURIER SERIES (DISCRETE-TIME SERIES EXPANSION)

Definition 05-9: Let $\{f[\ell]\} \in \ell_1(\mathrm{Z})$ be a N-periodic discrete-time sequence for some $N \in \mathrm{N}$, i.e., $f[n+N] = f[n]\ \forall n \in \mathrm{Z}$, then the discrete time Fourier series representation of $\{f[\ell]\}$ is given by:

$$\hat{f}[k] \equiv \sum_{\ell=0}^{N-1} W_N^{\ell k} f[\ell] \qquad \text{for any } k \in \mathrm{Z} \qquad \textbf{Analysis}$$

The inverse discrete time Fourier series representation of $\{\hat{f}[k]\}$ is given by:

$$f[\ell] \equiv \tfrac{1}{N} \sum_{k=0}^{N-1} W_N^{-\ell k} \hat{f}[k] \qquad \text{for any } \ell \in \mathrm{Z} \qquad \textbf{Synthesis}$$

where $W_N = \exp\left(-i \tfrac{2\pi}{N}\right)$ for any $N \in \mathrm{N}$. ♦

Properties of Discrete-Time Fourier Series: The following properties hold for $\forall f, g \in \ell_1(\mathrm{Z}); \forall \alpha, \beta \in \mathrm{C}$:

Linearity: $\alpha f[\ell] + \beta g[\ell] \leftrightarrow \alpha \hat{f}[k] + \beta \hat{g}[k]$

Shifting: $f[\ell - \ell_o] \leftrightarrow W_N^{\ell_o k} \hat{f}[k]$ and $W_N^{-\ell k_o} f[\ell] \leftrightarrow \hat{f}[k - k_o]$

Convolution: $f[\ell] * g[\ell] \equiv \sum_{m \in \mathrm{Z}} f[\ell - m] g[m] = \sum_{m \in \mathrm{Z}} f_o[(\ell - m) \bmod N] g_o[m]$ where $f_o[\bullet]$ and $g_o[\bullet]$ are equal to one period of $f[\bullet]$ and $g[\bullet]$, respectively, i.e., $f_o[\ell] = \begin{cases} f_o[\ell] \text{ for } \ell = 0,1,2,\cdots,N-1 \\ 0 \quad \text{otherwise} \end{cases}$, and similarly for $g_o[\ell]$. Then, $f[\ell] * g[\ell] \leftrightarrow \hat{f}[k]\hat{g}[k]$ and $f[\ell] g[\ell] \leftrightarrow \tfrac{1}{N} \hat{f}[k] * \hat{g}[k]$.

Parseval Equality:
$$\begin{cases} \langle f, g \rangle \equiv \sum_{\ell \in \mathrm{Z}} \bar{f}[\ell] g[\ell] = \tfrac{1}{N} \sum_{k \in \mathrm{Z}} \bar{\hat{f}}[k] \hat{g}[k] = \langle \hat{f}, \hat{g} \rangle \\ \|f\|^2 \equiv \sum_{\ell \in \mathrm{Z}} |f[\ell]|^2 = \tfrac{1}{N} \sum_{k \in \mathrm{Z}} |\hat{f}[k]|^2 = \|\hat{f}\|^2 \end{cases}$$
♦

Table I. Fourier Transforms with various combinations of time and frequency variables

Transform	Time	Freq. (hertz)	Analysis	Synthesis	Duality
(i) Continuous-time Fourier Transform (CTFT)	C	C	$\hat{f}(\xi) = \int_{\mathbb{R}} dt\, e^{-i2\pi \xi t} f(t)$	$f(t) = \int_{\mathbb{R}} d\xi\, e^{i2\pi \xi t} \hat{f}(\xi)$	Self-dual
(ii) Continuous-time Fourier Series (CTFS)	C P	D	$\hat{f}[k] = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} dt\, e^{-i2\pi k \frac{t}{T}} f(t)$	$f(t) = \sum_{k \in \mathbb{Z}} e^{i2\pi k \frac{t}{T}} \hat{f}[k]$	Dual with DTFT
(iii) Discrete-time Fourier Transform (DTFT)	D	C P	$\hat{f}(e^{i2\pi\xi}) = \sum_{\ell \in \mathbb{Z}} e^{-i2\pi \xi T \ell} f[\ell]$	$f[\ell] = \frac{1}{T} \int_{-\frac{1}{2}}^{\frac{1}{2}} d\xi\, e^{i2\pi \xi T \ell} \hat{f}(e^{i2\pi\xi})$	dual with CTFS
(iv) Discrete-time Fourier Series (DTFS)	D P	D P	$\hat{f}[k] = \sum_{\ell=0}^{N-1} e^{-i2\pi \ell k/N} f[\ell]$	$f[\ell] = \frac{1}{N} \sum_{\ell=0}^{N-1} e^{i2\pi \ell k/n} \hat{f}[k]$	Self-dual

C Continuous
D Discrete
P Periodic
S Series
T Time

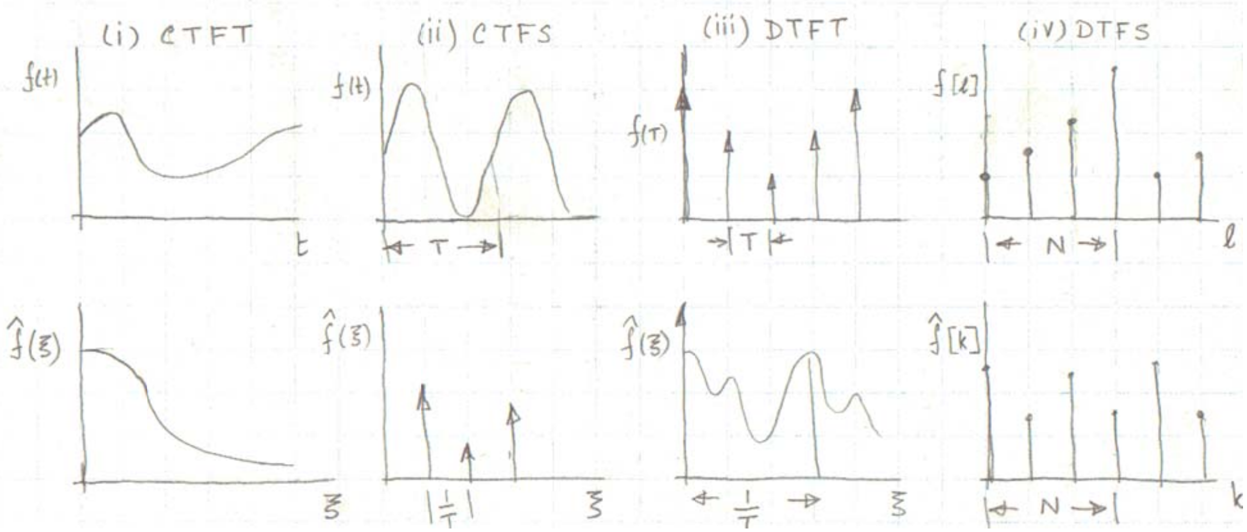

Table II. Various Fourier transforms with restrictions on the signals involved.
(Either the signal is of finite length or the Fourier transform is band-limited)

Transform	Time	Frequency (hertz)	Equivalence	Duality
(i) Fourier Transform of band-limited signals (BL-CTFT)	can be sampled	$\left[-\frac{1}{2T}, \frac{1}{2T}\right)$	sample time periodize frequency	dual with FL-CTFT
(ii) Fourier Transform of finite-length signals (FL-CTFT)	$[0, T)$	can be sampled	periodize time sample frequency	dual with BL-CTFT
(iii) Fourier Series of band-limited periodic signals (BL-CTFS)	periodic can be sampled	finite number of Fourier coeff.	sample time Finite Fourier series in time	dual with FL-DTFT
(iv) Discrete-time Fourier Transform of finite-length sequence (FL-DTFT)	finite number of samples	periodic can be sampled	sample frequency finite Fourier series in frequency	dual with BL-CTFS