Fall 2009

ME 550. Foundations of Engineering Systems Analysis

Take_Home Examination #1 **Due Date: September 18, 2009**

Problem #1. Let $f: X \to Y$. Let $A \subseteq X$ and $B \subseteq Y$.

(i) Show that $A \subseteq f^{-1}[f[A]]$. Cite an example to show that $A \subset f^{-1}[f[A]]$ is possible.

(ii) Show that $f[f^{-1}[B]] \subseteq B$. Cite an example to show that $f[f^{-1}[B]] = B$ is possible. **10 pts**

Problem #2. Consider the difference equation

$$x_{k+1} = \frac{1}{2}\left(x_k + \frac{2}{x_k}\right) \text{ with } x_1 = 1$$

Is $x_k \in \mathbf{Q} \ \forall k \in \mathbf{N}$?

If so, is the sequence $\{x_k\}$ a Cauchy sequence in $\mathbf{Q}$?

If so, does the sequence $\{x_k\}$ converge in $\mathbf{Q}$?

If not, where does $\{x_k\}$ converge in $\Re$? **10 pts**

Problem #3. Let $C^\infty[0,1]$ be the set of all real-valued infinitely differentiable functions on $[0,1]$. Let $D: C^\infty[0,1] \to C^\infty[0,1]$ be defined as: $Dx = \frac{dx}{dt}$. Let a metric on $C^\infty[0,1]$ be defined as:

$$d_\infty(x,y) = \underset{t \in [0,1]}{Max} \{|x(t) - y(t)|\}$$

Define another metric $\tilde{d}$ on $C^\infty[0,1]$ as $\tilde{d}(x,y) = d_\infty(x,y) + d_\infty(Dx, Dy)$

(a) Show whether or not D is a continuous mapping of the metric space $\langle C^\infty[0,1], d_\infty \rangle$ into itself.

(b) Show whether or not D is a continuous mapping of the metric space $\langle C^\infty[0,1], \tilde{d} \rangle$ into $\langle C^\infty[0,1], d_\infty \rangle$.

20 pts

Problem #4. Let a metric on a non-empty set S be defined as: $\mathrm{d(x,y)} = \begin{cases} 0 \text{ if } \mathrm{x = y} \\ 1 \text{ if } \mathrm{x \neq y} \end{cases}$.

Show that the metric space $\langle \mathrm{S, d} \rangle$ is separable if and only if S is countable.

. [A metric space is called separable if it has a countable dense subset.] **10 pts**

Problem #5. Let $C_1([0,1])$ be the subspace of $L_1([0,1])$ consisting of all continuous functions.

[Note: $f \in L_1([0,1])$ implies that $\|f\|_{L_1} \equiv \int_0^1 |f| \, dt < \infty$]

Show that $C_1([0,1])$ is not a complete space by constructing a Cauchy sequence in $C_1([0,1])$ that converges to a discontinuous function. Sketch the above sequence of functions consisting of all continuous functions and the limit point which should be a discontinuous function. **10 pts**

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Problem #6. Consider the closed interval [0,1] in $\Re \equiv (-\infty, \infty)$. Delete the open middle one-third, $(\frac{1}{3}, \frac{2}{3})$, from [0,1]. Then, delete, the open middle one-third of each remaining interval from the previous operation, and continue in this way. What remains of the original set is called the Cantor set $\mathfrak{C}$. For example, $C_1 := [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$ and $C_2 := [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$. Then, we have $\mathfrak{C} = \bigcap_{k=1}^{\infty} C_k$.

(a) Show that $\mathfrak{C}$ is a perfect set.
(b) Show that the Lebesgue measure of $\mathfrak{C}$ is 0.
(c) Show that $\mathfrak{C}$ is an uncountable set.
(d) Is $\frac{1}{4} \in \mathfrak{C}$? Is $\frac{1}{72} \in \mathfrak{C}$? Justify your answer. [Hint: Express the numbers in $\mathfrak{C}$ in base 3.] **40 pts**