Fall 2009

ME 550. Foundations of Engineering Systems Analysis

Take_Home Examination #2 **Due Date: October 16, 2009**

Problem#1. For a complex inner product space, show that $\langle y, x\rangle = \frac{1}{4}\sum_{k=1}^{4}\left(\sqrt{-1}\right)^k \left\| x + \left(\sqrt{-1}\right)^k y\right\|^2$ **10 pts**

Problem#2. Consider the real polynomials $p:[0,\infty)\to\Re$. Find an equivalent orthonormal set of the sequence of polynomials $\{1, t, t^2, t^3, \cdots\}$ in the Hilbert space $L_2([0,\infty), \exp(-t))$ by using Gram-Schmidt orthonormalization. Comment on how this orthonormal set is related to the set of Laguerre polynomials. **20 pts**

Problem#3. Let $\Re$ be the real field and **C** be the complex field. A trigonometric polynomial is a function of the form: $f(t) = \sum_{k=1}^{n} a_k \exp(i\lambda_k t)$ where $n \in \mathbf{N}$; $a_k \in \mathbf{C}$; and $\lambda_k \in \Re$. The space TP of trigonometric polynomials is a vector space over **C** with respect to pointwise addition and multiplication. Show that the following expression defines an inner product for TP:

$$\langle f, g\rangle = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} dt\, \bar{f}(t) g(t)$$ **8 pts**

Problem#4. Consider the space $C^1[0,1]$ of continuously differentiable functions on [0,1] with values in **C**. Show that the following expression defines an inner product for $C^1[0,1]$:

$$\langle f, g\rangle = \int_0^1 dt \left(\bar{f}(t) g(t) + \bar{\dot{f}}(t)\dot{g}(t)\right)$$ **8 pts**

Problem #5. Find an inner product on the space of polynomials such that the corresponding norm is given by:

$$\sqrt{\int_{-1}^{1} dt \left(|t| |f(t)|^2 + 3|\dot{f}(t)|^2 \right)}$$ **8 pts**

Problem #6. Show that, for any continuously differentiable function f on $[-\pi, \pi)$,

$$\left|\int_{-\pi}^{\pi} dt \left(f(t)\cos(t) - \dot{f}(t)\sin(t)\right)\right| \le \sqrt{2\pi\left(\int_{-\pi}^{\pi} dt \left(|f(t)|^2 + |\dot{f}(t)|^2\right)\right)}$$ **8 pts**

Problem #7. Show that, for any polynomial f,

$$\left|\int_{-1}^{1} dt \left(|t|^3 f(t) + 6t\dot{f}(t)\right)\right| \le \sqrt{\frac{25}{3}\left(\int_{-1}^{1} dt \left(|t| |f(t)|^2 + 3|\dot{f}(t)|^2\right)\right)}$$ **8 pts**

Problem #8. Consider the Hilbert space $L_2([-\pi,\pi))$ and the set of functions $\left\{\frac{1}{\sqrt{2\pi}}, \frac{\cos nt}{\sqrt{\pi}}, \frac{\sin nt}{\sqrt{\pi}}\right\}$ for $n \in \mathbf{N}$. Let S and A be the subsets of symmetric and antisymmetric functions on $[-\pi,\pi)$, i.e., $S = \{f \in L_2([-\pi,\pi)) : f(t) = f(-t)\}$ and $A = \{f \in L_2([-\pi,\pi)) : f(t) = -f(-t)\}$

(a) Show that S and A are subspaces of $L_2([-\pi,\pi))$.

(b) Show that the set $\left\{\frac{1}{\sqrt{2\pi}}, \frac{\cos nt}{\sqrt{\pi}}, \frac{\sin nt}{\sqrt{\pi}}\right\}$ for $n \in \mathbf{N}$ forms a complete orthonormal basis of $L_2([-\pi,\pi))$.

(c) Find an orthonormal basis for each of S and A. Show that $L_2([-\pi,\pi)) = S \oplus A$. **30 pts**