Fall 2009

ME 550. Foundations of Engineering Systems Analysis

Take_Home Examination #4 **Due Date: December 04, 2009**

Problem #1. Let $H \in \Re^{\ell \times n}$ where $\ell > n$ and $\text{rank}(H) = n$. Let $V \in \Re^{(\ell - n)\times \ell}$ be constructed such that the rows of V form an orthonormal basis for the null space of H^T.

(i) Show that $V^T V = I_\ell - H[H^T H]^{-1} H^T$ where I_ℓ is the $(\ell \times \ell)$ identity matrix.

(ii) If $H = [1\ 1\ 1]^T$ and $H = [1\ 1\ 1\ 1]^T$, construct matrices V that satisfy the above conditions. **30 pts**

Problem #2. Consider a signal $f \in L_1(\Re) \cap L_2(\Re)$. Let us bandlimit the signal to increasing bandwidths by a sequence of projection operators $\{P^k : L_2(\Re) \to L_2(\Re)\}$ defined as:

$$\left(P^k \hat{f}\right)(\xi) = \begin{cases} \hat{f}(\xi) & \text{for } \xi \in [-k, k] \\ 0 & \text{for } \xi \notin [-k, k] \end{cases}$$

where the Fourier transform of $f(t)$ is given as: $\hat{f}(\xi) \equiv \int_{\Re} dt \exp(i 2\pi \xi t)\, f(t)$.

(i) Show that P_k strongly converges to I where I is the identity operator.

(ii) Show that P_k does not converge to I in the operator norm. **10 pts**

Problem #3. Let a time-invariant operator $T : \ell_2 \to \ell_2$ be defined as:

$T(\{x_1, x_2, x_3, \cdots, x_n, \cdots\}) = \{y_1, y_2, y_3, \cdots, y_n, \cdots\}$ where $y_k = \sum_{j=1}^{k} t_{kj} x_j$ and $\sum_{k=1}^{\infty} |t_{k1}|^2 < \infty$.

(i) Is T linear?
(ii) Is T invertible?
(iii) Is T bounded? If so, find $\|T\|$.
(iv) Is T unitary?
(v) Determine T^*. Are the above properties of T similar to those of T^*? **20 pts**

Problem #4. For each $x \in L_2[0,1]$, let $y(t) = T\,x(t)$ be the solution of a time-invariant (i.e., constant coefficient) linear differential equation in the following two cases:

(i) $\frac{dy}{dt} + a_0\, y(t) = x(t)$ with $y(0) = 0$. Determine the adjoint operator T^*.
Is it ever possible that $T = T^*$?.

(ii) $\frac{d^2 y}{dt^2} + a_1 \frac{dy}{dt} + a_0\, y(t) = x(t)$ with $y(0) = 0$ and $y(1) = 0$.

Determine the adjoint operator T^*. Is it ever possible that $T = T^*$?. **40 pts**