ME (MATH) 577 Stochastic Systems in Science and Engineering

Take-home Examination #1; Due Date: February 04, 2008

Problem#1. Let the random vectors $X \in \mathbf{R}^n$ and $Y \in \mathbf{R}^m$ be jointly Gaussian having the density function as:

$$f_{XY}(\theta,\varphi) = \left[(2\pi)^{(n+m)} \left\| \begin{bmatrix} P_{XX} & P_{XY} \\ P_{XY}^T & P_{YY} \end{bmatrix} \right\| \right]^{-1/2} \exp\left\{ -\frac{1}{2} \begin{bmatrix} \theta - \mu_X \\ \varphi - \mu_Y \end{bmatrix}^T \begin{bmatrix} P_{XX} & P_{XY} \\ P_{XY}^T & P_{YY} \end{bmatrix}^{-1} \begin{bmatrix} \theta - \mu_X \\ \varphi - \mu_Y \end{bmatrix} \right\}$$

where μ indicates the expected value of a random vector and P indicates a covariance matrix.

(a) Let $\begin{bmatrix} P_{XX} & P_{XY} \\ P_{XY}^T & P_{YY} \end{bmatrix}^{-1} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$. Show that

$A_{21}^T = A_{12} = -A_{11} P_{XY} P_{YY}^{-1} = -P_{XX}^{-1} P_{XY} A_{22}$;

$A_{11} = \left(P_{XX} - P_{XY} P_{YY}^{-1} P_{YX} \right)^{-1}$; and $A_{22} = \left(P_{YY} - P_{YX} P_{XX}^{-1} P_{XY} \right)^{-1}$.

Also show that $A_{11} = P_{XX}^{-1} + P_{XX}^{-1} P_{XY} A_{22} P_{XY}^T P_{XX}^{-1}$; and $A_{22} = P_{YY}^{-1} + P_{YY}^{-1} P_{XY}^T A_{11} P_{XY} P_{YY}^{-1}$.

[Hint: You may use the matrix inversion lemma $[I + AB]^{-1} = I - A[I + BA]^{-1} B$ where A and B need not be square].

(b) Show that X and Y are uncorrelated, i.e., $E\left\{ X Y^T \right\} = E\{X\} E\{Y\}^T$ if and only if X and Y are statistically independent, i.e., $f_{XY}(\theta,\varphi) = f_X(\theta) f_Y(\varphi)$.

(c) Let us construct a p-dimensional random vector Z = QX + SY + C where Q, S and C are constant matrices of compatible dimension. Show that Z is Gaussian. Find the mean and covariance matrix of Z.
[Hint: You may use the characteristic function].

(d) Show that X and Y are individually Gaussian.

(e) Let $Z = \begin{bmatrix} V \\ W \end{bmatrix}$ be a linear transformation of $\begin{bmatrix} X \\ Y \end{bmatrix}$ via a bijective mapping from $\mathbf{R}^{n+m}$ onto $\mathbf{R}^{n+m}$ such that V and W are mutually uncorrelated and W=Y. Find an expression for P_{ZZ} in terms of P_{XX}, P_{XY} and P_{YY}.

(f) Using the relationship for conditional density $f_{X|Y}(\theta|\varphi) = \frac{f_{XY}(\theta,\varphi)}{f_Y(\varphi)}$ and the above results, show that

$$f_{X|Y}(\theta|\varphi) = \left[(2\pi)^n \left| P_{X|Y} \right| \right]^{-\frac{1}{2}} \exp\left\{ -\frac{1}{2} \left(\theta - \mu_{X|Y=\varphi} \right)^T P_{X|Y}^{-1} \left(\theta - \mu_{X|Y=\varphi} \right) \right\}$$

where $\mu_{X|Y=\varphi} = \mu_X + P_{XY} P_{YY}^{-1} (\varphi - \mu_Y)$ and $P_{X|Y} = P_{XX} - P_{XY} P_{YY}^{-1} P_{XY}^T$. **50 pts**

Problem#2. Let X be a random variable defined on a given probability space, and let $g: \mathbf{R} \to \mathbf{R}$ be a Borel-measurable function. Let us define Y=g(X).

(a) Is Y a random variable? If so, is Y defined on the same probability space as X? Justify your answer.

(b) Show that if g(•) is monotonically increasing (i.e., $\theta_1 < \theta_2$ implies $g(\theta_1) \le g(\theta_2)$), then

$$F_{XY}(\theta,\varphi) = \begin{cases} F_X(\theta) & \text{if } \varphi \ge g(\theta) \\ F_Y(\varphi) & \text{if } \varphi < g(\theta) \end{cases}$$

(c) Find $F_{XY}(\bullet,\bullet)$ if g(•) is monotonically decreasing (i.e., $\theta_1 < \theta_2$ implies $g(\theta_1) \ge g(\theta_2)$). **10 pts**

[Continued on the next page]

[Continued from the previous page]

Problem#3.

(a) Let a sequence of events $\{A_n\}$ on an event space $(\mathbf{R}, \Re)$ be defined as follows:

$$A_n = \begin{cases} (\frac{1}{n}, 1] & \text{if } n \text{ is odd} \\ (-1, \frac{1}{n}] & \text{if } n \text{ is even} \end{cases}$$

Find lim sup A_n and lim inf A_n. Does this sequence have a limit? If so, identify the limit.

(b) Let the event space be $(\mathbf{R}^2, \Re^2)$ and A_n be the open ball with center at $\begin{bmatrix} \frac{(-1)^n}{n} \\ 0 \end{bmatrix}$ and radius 1.

Find lim sup A_n and lim inf A_n. Does this sequence have a limit? If so, identify the limit. **20 pts**

Problem#4. Let Ω be a countably infinite set and $\mathbf{E}$ be the algebra consisting of all finite subsets of Ω and their complements. Define a measure ν on $\mathbf{E}$ as follows:

$$\nu(E) = \begin{cases} 0 & \text{if } E \in \mathbf{E} \text{ is finite} \\ 1 & \text{if } E^c \in \mathbf{E} \text{ is finite} \end{cases}$$

(a) Show that ν is finitely additive but not countably additive on $\mathbf{E}$.

(b) Show that Ω is the limit on a strictly increasing sequence of sets $E_k \in \mathbf{E}$ with $\nu(E_k) = 0\ \forall k \in \mathbf{N}$ but $\nu(\Omega) = 1$.

20 pts