Spring 2008

ME (MATH) 577. Stochastic Systems for Science and Engineering

Take_Home Examination #2; Due Date: February 20, 2008

Problem 1: Verify whether an event space can be a countably infinite collection of subsets of the sample space.

Problem 2: Show that the Borel set $\mathfrak{R}$ contains all open intervals, closed intervals, right semi-open intervals, and singletons.

Problem 3: Show that a continuous function $g: R \to R$ is a Borel measurable function.

Problem 4: Show that an indicator function $I_A : \Omega \to \{0,1\}$ is a Borel measurable function.

Problem 5: Show that if $\{E_k\}$ be an monotonically increasing sequence of events on a probability space $\langle \Omega, E, P \rangle$, then $\lim_{k \to \infty} P(E_k) = P\left(\bigcup_{k=1}^{\infty} E_k \right)$.

Problem 6: Show that every probability measure is *sequentially continuous* in the sense that if $\{E_k\}$ is a convergent sequence of events on a probability space $\langle \Omega, E, P \rangle$, then $\lim_{k \to \infty} P(E_k) = P\left(\lim_{k \to \infty} E_k \right)$.

Problem 7: Given an arbitrary sequence $\{E_k\}$ of events on a probability space $\langle \Omega, E, P \rangle$, show that $\sum_{k=1}^{\infty} P(E_k) < \infty$ implies that $P(\limsup E_k) = 0$. (Borel-Cantelli Lemma)

Problem 8: Establish truth or falsity of the following statements: (i) every random variable is the limit of a sequence of simple random variables; (ii) every nonnegative random variable is the limit of an increasing sequence of simple nonnegative random variables.

Problem 9: Let $\{X_k\}$ and $\{Y_k\}$ be two increasing sequences of nonnegative simple random variables converging to the same limit X. Show that $\lim_{k \to \infty} E[X_k] = \lim_{k \to \infty} E[Y_k]$.

Problem 10: Define the following set function P on the event space $([0,1], \mathfrak{R}[0,1])$ as:

$$P(E) = \begin{cases} \frac{1}{2} & \text{if } 0 \in E \text{ or } 1 \in E \text{ but not both} \\ 1 & \text{if } 0 \in E \text{ and } 1 \in E \\ 0 & \text{otherwise} \end{cases}$$

Verify whether P is a probability measure.

Problem 11: Let E and F be two events on a probability space $\langle \Omega, E, P \rangle$. Prove that if $P(E) \geq 1 - \delta$ and $P(F) \geq 1 - \delta$, then $P(E \cap F) \geq 1 - 2\delta$. In other words, if two events have probability close to 1, then their intersection also has a probability close to 1.

Problem 12: The Cauchy probability density function (pdf) is defined on R as: $f_X(\theta) = \frac{1}{\pi(1+\theta^2)}$.

Evaluate E{X} and verify that Var{X} is unbounded.