# Chernoff Bound

Let $\mathfrak{z}$ be a continuous random variable having the density $f_{\mathfrak{z}}(\cdot)$. We derive the Chernoff bound on the tail probability $P[\mathfrak{z} \geq \alpha]$ where $\alpha$ is a prescribed real constant.

$$P[\mathfrak{z} \geq \alpha] = \int_{\alpha}^{\infty} dx\, f_{\mathfrak{z}}(x) = \int_{\mathbb{R}} dx\, f_{\mathfrak{z}}(x)\, \mathcal{U}(x-\alpha) \qquad (1)$$

where $\mathcal{U}(\theta) = \begin{cases} 1 & \text{for } \theta \geq 0 \\ 0 & \text{for } \theta < 0 \end{cases}$ is the standard step function.

Since $\exp\left(t(x-\alpha)\right) \geq 1$ for all $t \geq 0$ and all $x \geq \alpha$,

$$P[\mathfrak{z} \geq \alpha] \leq \int_{\mathbb{R}} dx\, f_{\mathfrak{z}}(x) \exp\left(t(x-\alpha)\right)$$

$$= \exp(-\alpha t)\, \Theta_{\mathfrak{z}}(t) \qquad (2)$$

where $\Theta_{\mathfrak{z}}(t) \triangleq \int_{\mathbb{R}} dx\, f_{\mathfrak{z}}(x)\, e^{tx}$ is the moment generating function.

The tightest bound occurs when the right hand of Eq. (2) is minimized wrt $t$. This is called Chernoff bound.

Example: Let $\mathfrak{z} \sim N(\mu, \sigma^2)$ and let $\alpha > \mu$

$$\Theta_{\mathfrak{z}}(t) = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{\mathbb{R}} dx\, \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \exp(tx)$$

$$= \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right)$$ by completing the square in the integrand

$$P[\mathfrak{z} \geq \alpha] \leq \exp(-\alpha t)\, \Theta_{\mathfrak{z}}(t) = \exp\left(-(\alpha-\mu)t + \frac{\sigma^2 t^2}{2}\right)$$

$$\frac{d}{dt}\left(\exp\left(-(\alpha-\mu)t + \frac{\sigma^2 t^2}{2}\right)\right) = 0 \Rightarrow t = \frac{\alpha-\mu}{\sigma^2}$$

Hence, Chernoff bound is: $P[\mathfrak{z} \geq \alpha] \leq \exp\left(-\frac{(\alpha-\mu)^2}{2\sigma^2}\right)$

Verify that the above bound is tighter than Chebyshev bound for $\alpha > \mu$.

# Chernoff Bound for Discrete Random Variables

$P[N=k]$ denoted as $P_N(k)$ and $u(n-k) \triangleq \begin{cases} 1 & \text{for } n \geq k \\ 0 & \text{for } n < k \end{cases}$

$$P[N \geq k] = \sum_{n=k}^{\infty} P_N(n) = \sum_{n=0}^{\infty} P_N(n)\, u(n-k)$$

$$\leq \sum_{n=0}^{\infty} P_N(n) \exp\big(t(n-k)\big) \quad \text{where } t > 0$$

$$= \exp(-tk) \sum_{n=0}^{\infty} P_N(n) \exp(t)$$

$$= \exp(-tk)\, \Theta_N(t)$$

where $\Theta_N(t) \triangleq \sum_{n=0}^{\infty} P_N(n) \exp(t)$ is the moment generating function

**Example:** Let $N(t)$ be a Poisson process with the parameter $\lambda > 0$. Compute the Chernoff bound.

The Poisson process is a continuous-time Markov chain defined as

$$P[N=k] = e^{-\lambda} \frac{\lambda^k}{k!} \quad \text{for } k \in \mathbb{N} \cup \{0\}$$

$$\Theta_N(t) = \exp\big(\lambda(e^t - 1)\big)$$

and $\frac{d}{dt}\left[e^{-tk} \exp\big(\lambda(e^t-1)\big)\right] = 0$ yields

$$t_{min} = \ln \frac{k}{\lambda} \Rightarrow \Theta_N(t)\Big|_{t=\ln\frac{k}{\lambda}} = \exp\left(\lambda\left(\frac{k}{\lambda} - 1\right)\right)$$

Therefore, $P[N \geq k] \leq \left(\frac{\lambda}{k}\right)^k \exp(k - \lambda)$

is the Chernoff bound for the Poisson process.