Introduction to Wavelets

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Organization of the Presentation

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- Unified Theory of WFT and CWT
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What are Wavelets?

Waves: Waves are oscillating functions of time or space or both. For example, Fourier analysis is wave analysis. It decomposes signals in terms of (orthogonal) basis functions. Fourier analysis is useful for decomposition of stationary (e.g., periodic) signals.

Wavelets: Wavelets are small waves (in French, *les ondellettes*). A wavelet has oscillating wave-like characteristics and its energy is concentrated in time over relatively small intervals. Wavelets are especially useful for simultaneous time and frequency analysis of transient, non-stationary (e.g., structurally time-varying) signals.

Wavelets provide mathematical microscopy of non-stationary signals

Historical Review of Wavelets

- 1807 Fourier: Orthogonal decomposition of periodic signals
- 1873 Du Bois: Divergence of Fourier Transform
- 1900 Lebesgue: Completion of the theory of integration
- 1910 Haar: An alternative orthonormal basis for signal decomposition
- 1946 Gabor: Window Fourier Transform (WFT)
- 1976 Crochiere, Webber & Flanagan: Quadrature Mirror Filter (QMF)
- 1984 Grossman & Morlet: Square integrable wavelets
- 1984 Smith & Barnell: Reconstruction filtering in subband coding
- 1985 Mintzer: Distorsion-free two-band filter banks
- 1986 Vetterli: Filter banks for perfect reconstruction
- 1987 Vaidyanathan: M-channel maximally decimated quadrature filter bank
- 1988 Daubechies: Orthonormal bases of compactly supported wavelets
- 1989 Mallat: Multiresolution Analysis (MRA) of wavelets
- 1990 Meyer: Unified operator theory of wavelets
- 1990 Kaiser: Generalized wavelet transforms
- 1992 Coifman: Signal processing by wavelet analysis
- 1990's: Donoho: Wavelet analysis of statistical data
- 1995 Donoho & Johnstone: Wavelet shrinkage and thresholding
- 1996 Hall & Patil: Wavelets for estimation of smooth functions
- 1998 Resnikoff & Wells: Algebraic-geometric structure of wavelet analysis

Windowed Fourier Transform (WFT)

Let us start with Fourier transform of signals with compact support. For the space $L_2[-\frac{T}{2},\frac{T}{2}]$,

$$\text{For any } f \in L_2[-\tfrac{T}{2},\tfrac{T}{2}], \begin{cases} e_k^* f \equiv \langle e_k, f \rangle_T = \frac{1}{\sqrt{T}} \int_{-T/2}^{T/2} dt\, e^{-i 2\pi k \frac{t}{T}} f(t) & \textit{Analysis} \\ f(t) = \sum_{k=-\infty}^{\infty} e_k(t)\, e_k^* f = \frac{1}{\sqrt{T}} \sum_{k=-\infty}^{\infty} e^{i 2\pi k \frac{t}{T}} e_k^* f & \textit{Synthesis} \end{cases}$$

by choosing a (countable) orthonormal basis $e_k(t) \equiv \sqrt{\frac{1}{T}}\, e^{i 2\pi k \frac{t}{T}}$ for $k \in N$.

Defining $\xi_k \equiv k/T$ and $\Delta\xi_k \equiv \xi_{k+1} - \xi_k = 1/T$

$$\begin{cases} \hat{f}_T(\xi_k) \equiv \int_{-T/2}^{T/2} dt\ e^{-i 2\pi \xi_k t}\ f(t) \\ f(t) = \sum_{k=-\infty}^{\infty} \Delta\xi_k\, e^{i 2\pi \xi_k t}\, \hat{f}_T(\xi_k) \end{cases} \qquad \text{As } T \to \infty, \begin{cases} \hat{f}(\xi) \equiv \int_{-\infty}^{\infty} dt\, e^{-i 2\pi \xi t}\ f(t) \\ f(t) = \int_{-\infty}^{\infty} d\xi\, e^{i 2\pi \xi t}\ \hat{f}(\xi) \end{cases}$$

Construction of WFT

The signal $f \in L_2(\Re)$ is weighted by the window function $g \in L_2(\Re)$ with $\operatorname{supp} g \subseteq [-T, 0]$.

The window-weighted function f_t is obtained as:

$$f_t(\tau) \equiv \bar{g}(\tau - t)\, f(\tau) \text{ where } \bar{g}(\bullet) \text{ is the complex conjugate of } g(\bullet).$$

We define Windowed Fourier Transform (**WFT**) $\tilde{f}(\xi, t)$ of the signal $f(t)$ as:

$$\tilde{f}(\xi, t) \equiv \hat{f}_t(\xi) = \int_{\Re} d\tau\, e^{-i 2\pi \xi \tau} f_t(\tau)$$

$$= \int_{\Re} d\tau\, e^{-i 2\pi \xi \tau} \bar{g}(\tau - t)\, f(\tau)$$

Defining $g_{\xi,t} : \Re \to \boldsymbol{C}$ as $g_{\xi,t}(\tau) \equiv e^{i 2\pi \xi \tau} g(\tau - t) \Rightarrow \left\| g_{\xi,t} \right\|_{L_2} = \left\| g \right\|_{L_2}$, WFT can be expressed as:

$$\tilde{f}(x, t) = \left\langle g_{x,t}, f \right\rangle \equiv g^*_{x,t} f \Rightarrow \left| \tilde{f}(x, t) \right| \leq \left\| g \right\| \left\| f \right\| \text{ by Cauchy-Schwarz inequality}$$

where the linear bounded functional $g^*_{\xi,t} : L_2(\Re) \to \boldsymbol{C}$ is called the adjoint of $g_{\xi,t}$.

Note: WFT converts a 1-dimensional function $f(t)$ to a 2-dimensional function $\tilde{f}(\xi, t)$ without changing its total energy. The space of WFT becomes: $F \equiv \left\{ \tilde{f}(\bullet,\bullet) : f \in L_2(\Re) \right\} \subset L_2(\Re \times \Re)$.

Signal Processing in Time-Frequency Domain

Heisenberg Uncertainty Principle: Let $f \in L_2(\mathfrak{R})$ be a unit energy signal and let $\hat{f}(\bullet)$ be its Fourier transform. By appropriate time translation and frequency modulation, let $f(t)$ and $\hat{f}(\xi)$ be centered around $t = 0$ and $\xi = 0$, respectively.

The time width of $f(t)$ is defined as: $T \equiv \sqrt{\int_{\mathfrak{R}} dt \left| t\, f(t) \right|^2}$

The frequency width of $\hat{f}(\xi)$ is defined as: $W \equiv \sqrt{\int_{\mathfrak{R}} d\xi \left| \xi\, \hat{f}(\xi) \right|^2}$

If $f(t) \sim o\left(\sqrt{t}\right)$, then $(\Omega T) \geq \frac{1}{4\pi}$ and equality holds only for Gaussian $f(t) \equiv \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{t^2}{2\sigma^2}}$. ♦

Expressing WFT as: $\tilde{f}(\xi,t) \equiv \hat{f}_t(\xi) = \int_{\mathfrak{R}} d\tau\, e^{-i2\pi\xi t} f_t(\tau) = \int_{\mathfrak{R}} d\tau\, e^{-i2\pi\xi t} \overline{g}(\tau - t)\, f(\tau)$, it follows that:

$$\tilde{f}(\xi,t) = \left\langle \hat{g}_{\xi,t}, \hat{f} \right\rangle = e^{-i2\pi\xi t} \left(\overline{\hat{g}}(\zeta - \xi)\, \hat{f}(\zeta) \right)^{\vee}(t)$$

Note: Inverse Fourier transform of the frequency-localized signal $\overline{\hat{g}}(\zeta - \xi)\, \hat{f}(\zeta)$ is multiplied by $e^{-i2\pi\xi t}$ which is a modulation due to translation of $\hat{g}$ by ξ in the frequency domain.

Key Theorems on WFT

Theorem 1: A function $h(\xi,t) \in F$, the space of WFT of square-integrable functions, if and only if $h \in L_2(\Re \times \Re)$ and, in addition, satisfies the following consistency condition:

$$h(\xi,t) = \iint_{\Re\times\Re} d\xi' dt' K(\xi,t|\xi',t')\, h(\xi',t')$$

where the reproducing kernel $K(\xi,t|\xi',t') \equiv g^*_{\xi,t}\, g^{\xi',t'} = \left\langle g_{\xi,t}, g^{\xi',t'} \right\rangle$

$$= \int_{\Re} d\tau\, \bar{g}_{\xi,t}(\tau) g^{\xi',t'}(\tau) = \|g\|_{L_2}^{-2} \int_{\Re} d\tau\, e^{-i2\pi(\xi-\xi')\tau}\, \bar{g}(\tau-t)\, g(\tau-t')$$

and $g^{\xi,t}(\tau) \equiv \|g\|_{L_2}^{-2} g_{\xi,t}(\tau) \;\; \forall \tau \in \Re$ ♦

Note: Resolution of Identity is: $\iint_{\Re\times\Re} d\xi\, dt\, g^{\xi,t} g^*_{\xi,t} = \|g\|_{L_2}^{-2} \iint_{\Re\times\Re} d\xi\, dt\, g_{\xi,t}\, g^*_{\xi,t} = I$

Theorem 2: Given an arbitrary function $h \in L_2(\Re \times \Re)$, let $f_h(\bullet) \equiv \iint_{\Re\times\Re} d\xi\, dt\, g_{\xi,t}(\bullet)\, h(\xi,t)$.

Then, (i) $f_h \in L_2(\Re)$; and

(ii) $\left\| h - \tilde{f} \right\|_{L_2(\Re\times\Re)} > \left\| h - \tilde{f}_h \right\|_{L_2(\Re\times\Re)} \; \forall f \in L_2(\Re)$ such that $\| f - f_h \|_{L_2(\Re)} > 0$. ♦

Continuous Wavelet Transform (CWT)

Consider the distribution $\delta_\varepsilon(t) \equiv \frac{1}{\pi}\left(\frac{\varepsilon}{t^2+\varepsilon^2}\right)$ for $\varepsilon > 0$ and let $f(\bullet)$ be continuous and bounded on $\Re$.

Following Gel'fand and Shilov (1964), we define $\delta(t) = \lim_{\varepsilon \to 0+} \delta_\varepsilon(t)$. Then,

$$\int_{\Re} dt\, f(t)\delta(t) = f(0) \text{ implying that } \hat{\delta}(\xi) \equiv \int_{\Re} dt\, e^{-i2\pi\xi t}\,\delta(t) = 1 \text{ and } \delta(t) \equiv \int_{\Re} d\xi\, e^{i2\pi\xi t}$$

Intuitively, $t = 0$ yields $\delta(t) = \infty$ Constructive Interference of exponentials of all frequencies

$t \neq 0$ yields $\delta(t) = 0$ Destructive Interference of exponentials of all frequencies

Fourier analysis attempts to compose **local behavior** of signals with **non-local exponentials** $e^{i2\pi\xi t}$. WFT only partially solves this problem by using $\tilde{f}(\xi, t)$ as a coefficient function to superimpose $g_{\xi,t}(\tau)$ for reconstruction of the original signal $f(t)$. Note that $g_{\xi,t}(\tau)$ has a compact support $[t - T, t]$.

Therefore, WFT introduces a ***scale*** (i.e., the window width T) into analysis and synthesis of signals. Features with time scale much shorter than T can be synthesized with $g_{\xi,t}$'s for closely spaced time instants t and different frequencies ξ. Similarly, features with time scale much longer than T can be synthesized with $g_{\xi,t}$'s for closely spaced frequencies ξ and different time instants t.

Wavelets provide ***scale-independence*** for analysis and synthesis of signals.

Construction of CWT

Let us start with a (complex-valued) window function, $\psi_1 \in L_2(\Re)$, which is called the ***mother wavelet***. For an arbitrary $p \in [0,\infty)$ and ***scale*** $s \in \Re \setminus \{0\}$, we define the wavelet function:

$$\psi_s(\tau) \equiv |s|^{-p}\,\psi_1\!\left(\tfrac{\tau}{s}\right) \text{ such that } \psi_s(\tau)\big|_{s=1} = \psi_1(\tau)$$

where $s > 1$ implies signal stretching; $s \in (0,1)$ implies signal compression; and $s < 0$ represents signal reflection; and if $p = 0.5$, then $\|\psi_s\|_{L_2} = \|\psi_1\|_{L_2}\ \forall s$.

Time localization is achieved through translated versions of ψ_s. If $\psi_1(\tau)$ is supported on an interval of length T near $\tau = 0$, then $\psi_s(\tau)$ is supported on an interval of length $|s|T$ near $\tau = 0$, and

$$\psi_{s,t}(\tau) \equiv \psi_s(\tau - t) = |s|^{-p}\,\psi_1\!\left(\tfrac{\tau - t}{s}\right)$$

supported on an interval of length $|s|T$ near $\tau = t$

$$\|\psi_{s,t}\|_{L_2}^2 = |s|^{1-2p}\,\|\psi_1\|_{L_2}^2$$

Continuous wavelets $\psi_{s,t}$, generated by the mother ψ_1, are conceptually similar to $g_{\xi,t}$ of WFT. Now, we define the continuous wavelet transform (**CWT**) of a signal $f \in L_2(\Re)$ as:

$$\tilde{f}(s,t) \equiv \int_{-\infty}^{\infty} d\tau\, \psi_{s,t}(\tau)\, f(\tau) = \langle \psi_{s,t}, f \rangle = \psi_{s,t}^{*} f \quad \text{and} \quad \langle \psi_{s,t}, f \rangle = \langle \hat{\psi}_{s,t}, \hat{f} \rangle$$

Signal Reconstruction

The Fourier transform of the wavelet functions: $\hat{\psi}_{s,t}(\xi) = |s|^{1-p} e^{-i2\pi\xi t} \hat{\psi}_1(s\xi)$, yields:

$$\tilde{f}(s,t) = |s|^{1-p} \int_{-\infty}^{\infty} d\xi \, e^{i2\pi\xi t} \, \overline{\hat{\psi}_1}(s\xi) \, \hat{f}(\xi) = |s|^{1-p} \left(\overline{\hat{\psi}_1}(s\xi) \, \hat{f}(\xi) \right)^{\vee} (t)$$

and hence $|s|^{1-p} \overline{\hat{\psi}_1}(s\xi) \, \hat{f}(\xi) = \int_{-\infty}^{\infty} dt \, e^{-i2\pi\xi t} \, \tilde{f}(s,t)$

Note that the inverse of $\overline{\hat{\psi}_1}(s\xi)$ may not be well-defined over the frequency range of interest.

Let us generate a piecewise constant weighted average of $\hat{\psi}_1(s\xi)$ defined as:

$$C_+ \equiv \int_0^{\infty} \frac{ds}{s} |\hat{\psi}_1(s\xi)|^2 = \int_0^{\infty} \frac{d\theta}{\theta} |\hat{\psi}_1(\theta)|^2 \quad \text{for } \xi > 0$$

$$C_- \equiv \int_0^{\infty} \frac{ds}{s} |\hat{\psi}_1(s\xi)|^2 = \int_0^{\infty} \frac{d\theta}{\theta} |\hat{\psi}_1(\theta)|^2 \quad \text{for } \xi < 0$$

which defines the admissibility condition for the ***mother wavelet*** ψ_1 for all $\xi \neq 0$ (hence a.e. on $\Re$) iff

$$0 < C_{\pm} \equiv \int_0^{\infty} \frac{d\theta}{\theta} |\hat{\psi}_1(\pm\theta)|^2 < \infty$$

We define the ***reciprocal mother wavelet*** as:

$$\psi^1(\tau) = C_-^{-1} \int_{-\infty}^{0} d\xi \, e^{i2\pi\xi\tau} \hat{\psi}_1(\xi) + C_+^{-1} \int_0^{\infty} d\xi \, e^{i2\pi\xi\tau} \hat{\psi}_1(\xi) \equiv C_-^{-1} \psi_1^-(\tau) + C_+^{-1} \psi_1^+(\tau)$$

Key Theorems on CWT

Theorem 3: Let the mother wavelet ψ_1 be admissible and let $\{\psi^{s,t}\}$ be the family spawned by the reciprocal mother ψ^1. Then, $f \in L_2(\Re)$ can be reconstructed from its CWT $\tilde{f}(s,t) = \psi^*_{s,t} f$ by:

$$f(\bullet) = \int_{\Re_+} ds\, s^{2p-3} \int_{\Re} dt\, \psi^{s,t}(\bullet)\, \tilde{f}(s,t) \quad \text{where } \Re_+ \equiv (0,\infty)$$

$$\Rightarrow \quad \int_{\Re_+} ds\, s^{2p-3} \int_{\Re} dt\, \psi^{s,t}\, \psi^*_{s,t} = I \qquad \text{Resolution of Identity in } L_2(\Re). \qquad \blacklozenge$$

Remark: ψ_1 is real-valued, then $\hat{\psi}_1(-\xi) = \overline{\hat{\psi}_1(\xi)}$ implying that $C_- = C_+ = C/2 \Rightarrow \psi^1 = \frac{2}{C}\psi_1$. ♦

Theorem 4: Let the mother wavelet ψ_1 be admissible in the sense that $0 < C < \infty$ where

$$C \equiv \int_{\Re} \frac{d\theta}{|\theta|} \left|\hat{\psi}_1(\theta)\right|^2 = C_- + C_+$$

Then, $f \in L_2(\Re)$ can be reconstructed from its CWT $\tilde{f}(s,t) = \psi^*_{s,t} f$ by:

$$f(\bullet) = C^{-1} \int_{\Re} ds\, |s|^{2p-3} \int_{\Re} dt\, \psi_{s,t}(\bullet)\, \tilde{f}(s,t)$$

$$\Rightarrow \quad C^{-1} \iint_{\Re\times\Re} ds\, |s|^{2p-3} dt\, \psi_{s,t}\psi^*_{s,t} = I \qquad \text{Resolution of identity in } L_2(\Re). \qquad \blacklozenge$$

Key Theorems on CWT (continued)

Definition 1: A function $f \in L_2(\Re)$ is called $\begin{cases} \textit{upper analytic} \text{ if } \hat{f}(\xi) = 0 \ \ \forall \xi < 0 \\ \textit{lower analytic} \text{ if } \hat{f}(\xi) = 0 \ \ \forall \xi > 0 \end{cases}$ ♦

Remark: A function $f \in L_2(\Re)$ can be uniquely expressed as the sum $f = f_+ + f_-$ where

$$f_+(\tau) \equiv \int_0^\infty d\xi\, e^{i2\pi\xi\tau} \hat{f}(\xi) \text{ and } f_-(\tau) \equiv \int_{-\infty}^0 d\xi\, e^{i2\pi\xi\tau} \hat{f}(\xi)$$

Therefore, $\langle f_+, f_- \rangle = 0$ and the upper and lower analytic functions respectively form two subspaces $L_2^+(\Re)$ and $L_2^-(\Re)$ that are mutually orthogonal and complement to each other in $L_2(\Re)$. ♦

Theorem 5: Let $f \in L_2(\Re)$ and let the mother wavelet ψ_1 be upper analytic. Then, the CWT

$$\tilde{f}(s,t) = \begin{cases} \tilde{f}_+(s,t) & \text{if } s > 0 \\ \tilde{f}_-(s,t) & \text{if } s < 0 \end{cases}$$

Furthermore, $f_\pm(\bullet) = C^{-1} \iint_{\Re_\pm \times \Re} ds\, |s|^{2p-3} dt\, \psi_{s,t}(\bullet)\, \tilde{f}(s,t)$

and the orthogonal projections of $L_2(\Re)$ onto the subspaces $L_2^\pm(\Re)$ are given by:

$$P^\pm \equiv C^{-1} \iint_{\Re_\pm \times \Re} ds\, |s|^{2p-3} dt\, \psi_{s,t} \psi_{s,t}^*$$ ♦

Comparison of WFT and CWT

Windowed Fourier Transform (WFT)	Continuous Wavelet Transform (CWT)
$f \in L_2(\Re);\ \ g \in L_2(\Re)$ and $\mathrm{supp}\ g \subset [-T, 0]$	$f \in L_2(\Re);\ \ \psi_1 \in L_2(\Re)$; for $s \neq 0$ and $p \geq 0$
$g_{\xi,t}(\tau) \equiv e^{i2\pi\xi\tau} g(\tau - t)$ Window weight family	$\psi_{s,t}(\tau) \equiv s^{-p}\, \psi_1\left(\frac{\tau - t}{s}\right)$ Wavelet family
$g^{\xi,t}(\tau) \equiv \lVert g \rVert_{L_2}^{-2}\ g_{\xi,t}(\tau)$ Reciprocal family	$\psi^{s,t}(\tau) \equiv s^{-p}\, \psi^1\left(\frac{\tau - t}{s}\right)$ Reciprocal family
	where $\begin{cases} \psi^1(\tau) \equiv \dfrac{\psi_1^-(\tau)}{C_-} + \dfrac{\psi_1^+(\tau)}{C_+} \\ C_\pm \equiv \displaystyle\int_{\Re_+} \frac{d\theta}{\theta} \left\lvert \hat{\psi}_1(\pm\theta) \right\rvert \in (0, \infty) \end{cases}$
$\tilde{f}(x,t) = g^*_{x,t} f \equiv \langle g_{x,t}, f \rangle$ Analysis	$\tilde{f}(s,t) = \psi^*_{s,t} f \equiv \langle \psi_{s,t}, f \rangle$ Analysis
$f(\bullet) = \displaystyle\iint_{\Re\times\Re} d\xi\, dt\, g^{\xi,t}(\bullet)\, \tilde{f}(\xi,t)$ Synthesis	$f(\bullet) = \displaystyle\iint_{\Re_+\times\Re} ds\, s^{2p-3} dt\ \psi^{s,t}(\bullet)\, \tilde{f}(s,t)$ Synthesis
$\displaystyle\iint_{\Re\times\Re} d\xi\, dt\, g^{\xi,t} g^*_{\xi,t} = I$ Resolution of Identity	$\displaystyle\iint_{\Re_+\times\Re} ds\, s^{2p-3} dt\, \psi^{s,t} \psi^*_{s,t} = I$ Resolution of Identity
$K(\xi,t \vert \xi',t') \equiv g^*_{\xi,t}\, g^{\xi',t'}$ Reproducing kernel	$K(s,t \vert s',t') \equiv \psi^*_{s,t}\, \psi^{s',t'}$ Reproducing kernel
$\tilde{f}(\xi,t) = \displaystyle\iint_{\Re\times\Re} d\xi\, dt\ K(\xi,t \vert \xi',t')\, \tilde{f}(\xi',t')$	$\tilde{f}(s,t) = \displaystyle\iint_{\Re_+\times\Re} ds\, s^{2p-3} dt\ K(s,t \vert s',t')\, \tilde{f}(s',t')$

Unified Theory of WFT and CWT

For ***analysis*** (i.e., $f(\bullet) \mapsto \tilde{f}(*,\circ)$), we require a Hilbert space H of signals, a set M of labels, and a family of vectors $h_m \in H$ labeled by $m \in M$. Then, the requirements of WFT and CWT are:

WFT: $H = L_2(\Re)$; $M = \Re \times \Re$; $m = (\xi, t)$; and $h_m = g_{\xi,t}$ (also $h^m = g^{\xi,t}$)

CWT: $H = L_2(\Re)$; $M = \Re_+ \times \Re$; $m = (s,t)$; and $h_m = \psi_{s,t}$ (also $h^m = \psi^{s,t}$)

For ***synthesis*** (i.e., $\tilde{f}(*,\circ) \mapsto f(\bullet)$), we construct a family of vectors $\{h^m\}$ in H, which are ***reciprocals*** of $\{h_m\}$ in an appropriate sense. Then, we integrate over the set M of labels to reconstruct the signal f as a superposition of the vectors h^m. This requires a σ- finite measure Π on the measurable space M.

The measure spaces for WFT and CWT are as follows:

WFT: $M = \Re \times \Re$; $\Pi(A) \equiv$ area (i.e., Lebesgue measure) of any measurable set A in M.

Therefore, for every $\Pi-$measurable $\tilde{f} : M \to \boldsymbol{C}$, $\int_M d\Pi(m)\, \tilde{f}(m) = \iint_{\Re\times\Re} d\xi\, dt\, \tilde{f}(\xi,t)$

CWT: $M = \Re_+ \times \Re$; ; $\Pi(A) \equiv \iint_A ds\, s^{2p-3}\, dt$ for any measurable set A in M. Therefore,

for every $\Pi-$measurable $\tilde{f} : M \to \boldsymbol{C}$, $\int_M d\Pi(m)\, \tilde{f}(m) = \iint_{\Re_+\times\Re} ds\; s^{2p-3}\, dt\, \tilde{f}(\xi,t)$

Definition 2: Let H be a Hilbert space and let M be a measure space with measure Π. A generalized frame in H indexed by M is a family of vectors, called ***frame vectors***, $\{h_m \in H : m \in M\}$ such that

(a) $\forall f \in H$, $\tilde{f} : M \to \boldsymbol{C}$ defined by $\tilde{f}(m) \equiv \langle h_m, f \rangle_H$ is a Π-measurable function.

(b) $\exists$ a pair of ***frame bounds*** $0 < B_\ell \le B_u < \infty$ such that $\forall f \in H$, $B_\ell \|f\|_H^2 \le \|\tilde{f}\|_{L_2}^2 \le B_u \|f\|_H^2$.

Analyzing and Synthesizing Operators

- ***Given an arbitrary signal*** $g(m) \in L_2(\Pi)$***, whether there exist*** $f \in H$ ***such that*** $g = \tilde{f}$ ***?***

Answer: Find the range of the ***analyzing operator*** $T : H \to L_2(\Pi)$, i.e., find $\left\{\tilde{f} \in L_2(\Pi) : f \in H\right\}$.

- ***If*** $g = \tilde{f}$ ***for some*** $f \in H$***, then how do we reconstruct*** f ***?***

Answer: Find left inverse of T, i.e., the ***synthesizing operator*** $S : L_2(\Pi) \to H$ such that $ST = I$.

Both questions can be answered by finding a (two-sided) invertible metric operator G such that

$$G \equiv \int_M d\Pi(m)\, h_m h_m^* = I\,; \quad \left(B_\ell I \le G \le B_u I \Leftrightarrow B_u^{-1} I \le G^{-1} \le B_\ell^{-1} I\right)$$

and $\quad S = G^{-1} T^* \;\Rightarrow\; T^* T = G$

- **Reconstruction is achieved as:** $f = S\tilde{f}$
- **The range of** T **is obtained from the orthogonal projection operator** TS.

Summary of Pertinent Results

- $TS = TG^{-1}T^*$ is the orthogonal projection operator onto the range of T in $L_2(\Pi)$ *where* $G \equiv \int_M d\Pi(m)\, h_m h_m^*$ Construction of the reciprocal family becomes meaningful if $G \neq I$
- Adjoint $T^*: L_2(\Pi) \rightarrow H$ of analyzing operator T yields: $T^* g = \int_M d\Pi(m)\, h_m\, g(m)$ (weak sense)
- The synthesizing operator $S = G^{-1}T^* : L_2(\Pi) \rightarrow H$ yields: $Sg = \int_M d\Pi(m)\, h^m g(m)$.
- The signal $f \in H$ is reconstructed from $\tilde{f} \in L_2(\Pi)$ as: $f = S\tilde{f} = \int_M d\Pi(m)\, h^m \tilde{f}(m)$.
- The orthogonal projection $P: L_2(\Pi) \rightarrow L_2(\Pi)$ onto the range of T is given by:

  $Pg(m) = \int_M d\Pi(m')\, K(m|m')\, g(m')$ with reproducing kernel $K(m|m') \equiv \langle h_m, h^{m'} \rangle = \langle h_m, G^{-1} h_{m'} \rangle$.

  Specifically, $g \in L_2(\Pi)$ belongs to $\{\tilde{f} \in L_2(\Pi) : f \in H\}$ iff it satisfies the consistency condition:

  $g(m) = \int_M d\Pi(m')\, K(m|m')\, g(m')$. Resolution of Identity is: $\int_M d\Pi(m)\, h^m h_m^* = I$.
- (Least-Squares Approximation): Let $g \in L_2(\Pi)$ be an arbitrary signal, *not necessarily* belonging to $\{\tilde{f} \in L_2(\Pi) : f \in H\}$. The signal that minimizes the *error* $\left\| g - \tilde{f} \right\|_{L_2}^2 \equiv \int_M d\Pi(m) \left| g(m) - f(m) \right|^2$ is uniquely determined as: $f = Sg = \int_M d\Pi(m)\, h^m g(m)$.

Discrete Wavelet Transform (DWT)

Discretization of the domains of frequency (scale) and/or time changes the measure of the set M.
If both frequency (scale) and time are discretized, then we assume that

- *Every* subset of the countable set M is a measurable set
- *Every* $g : M \to \mathbf{C}$ is a measurable function.
- The integral over M becomes a sum, i.e., $\int_M d\Pi(m)\, g(m) = \sum_{m \in M} \pi_m\, g(m)$ with $\pi_m \in (0, \infty)$.
- $\exists$ ***frame bounds*** $0 < B_\ell \le B_u < \infty$ such that $\forall f \in H$, $B_\ell \|f\|_H^2 \le \sum_{m \in M} \pi_m \left\| \tilde{f}(m) \right\|^2 \le B_u \|f\|_H^2$.

Theorem 6: A discrete frame is a basis if and only if its reproducing kernel is given by:

$$K(\ell | m) \equiv h_\ell^* h^m = \pi_m^{-1} \delta_\ell^m \quad \forall \ell, m \in M$$

In that case, $\{\pi_m h^m\}$ is a basis biorthogonal to $\{h_m\}$.

Theorem 7: Let H_M be a normalized discrete frame (i.e., $\|h_m\| = 1$) with constant $\pi_m = \gamma \ \forall m \in M$.

Then, the upper frame bound $B_u \ge \gamma$ and the equality holds iff H_M is an orthonormal basis. In that case, $G = \gamma I$ and H_M is tight (i.e., $B_u = B_\ell$). Furthermore, H_M is self-reciprocal (i.e., $G = I$) iff H_M is an orthonormal basis.

Wavelets and Probability Density

The CWT $\tilde{f}(s,t)$ is interpreted as *details* of the signal f at the scale s and time t.

Expand this concept in terms of a probability density function $\varphi(\tau)$ with zero mean and unit variance, i.e.,

$$\int_{\Re} d\tau\varphi(\tau) = 1; \quad \int_{\Re} d\tau\big(\tau\varphi(\tau)\big) = 0; \text{ and } \int_{\Re} d\tau\big(\tau^2\varphi(\tau)\big) = 1.$$

Assuming that $\varphi(\tau)$ is at least n times differentiable and that its $(n-1)^{th}$ derivative satisfies:

$$\lim_{\tau\to\pm\infty} \varphi^{(n-1)}(\tau) = 0$$

Let $\psi^n(\tau) \equiv (-1)^n \varphi^{(n)}(\tau)$ so that $\int_{\Re} d\tau\psi^n(\tau)=0$ satisfies the admissibility condition of a wavelet.

For $s \neq 0$ and $t \in \Re$, let $\varphi_{s,t}(\tau) \equiv |s|^{-1}\varphi\left(\frac{\tau-t}{s}\right)$ and $\psi^n_{s,t}(\tau) \equiv |s|^{-1}\psi^n\left(\frac{\tau-t}{s}\right)$.

Then, the scaling function $\varphi_{s,t}$ is a probability distribution function with mean t and variance s^2,

and $\psi^n_{s,t}$ is the wavelet family of ψ^n if the wavelet parameter $p = 1$.

Note that the local average $\bar{f}(s,t) = \varphi^*_{s,t} f$ and the details $\tilde{f}(s,t) = (\psi^n_{s,t})^* f$.

For Gaussian distribution $\varphi(\tau) = \frac{1}{\sqrt{2\pi}}\exp(-\frac{\tau^2}{2})$, the generated wavelets for $n = 1$ and $n = 2$ are:

$$\psi^1(\tau) = -\varphi^{(1)}(\tau) = \frac{\tau}{\sqrt{2\pi}}\exp(-\frac{\tau^2}{2}) \text{ and } \psi^2(\tau) = \varphi^{(2)}(\tau) = \frac{(\tau^2-1)}{\sqrt{2\pi}}\exp(-\frac{\tau^2}{2})$$