

Introduction to Multiresolution Analysis of Wavelets

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Organization of the Presentation

- What is Multiresolution Analysis (MRA)?
- Scaling Functions and Average Spaces
- Wavelet Functions and Detail Spaces
- An Example of Scaling and Wavelet Functions
- Implementation of MRA as Filter Banks
- Subband Filtering Scheme
- Signal Analysis and Synthesis Using QMFs
- Examples of MRA Using Matlab Demo

What is Multiresolution Analysis?

- Let $f^0 \equiv \{f_k^0\}$ be a sequence of sampled signals at the intervals $\Delta t \equiv \tau > 0$, i.e., scale τ .
With no loss of generality, we set $\tau = 1$.
- Split f^0 into: **average** f^1 at coarse scale 2τ and **details** d^1 at fine scale $\tau \Rightarrow f^0 = f^1 + d^1$
- The process goes on recursively to yield **a progressively coarser sequences** f^0, f^1, f^2, \dots
with details d^0, d^1, d^2, \dots removed at every scale $\Delta t \equiv 2^m \tau$ in f^m and d^{m+1} .

This yields: $f^0 = d^1 + d^2 + \dots + d^N + f^N$ for some $N \in \mathbb{Z}$

- An **orthonormal base** is created by having the **scale** $s = 2$ in the recursive extraction of details.
- The **time translation and dilation operators** $T, D : L_2(\mathbb{R}) \rightarrow L_2(\mathbb{R})$ are defined, respectively, by:

$$\left. \begin{aligned} Tf(t) = f(t-1) &\Rightarrow (T^k f)(t) = f(t-k) \\ Df(t) = \frac{1}{\sqrt{2}} f\left(\frac{t}{2}\right) &\Rightarrow (D^k f)(t) = \frac{1}{\sqrt{2^k}} f(2^{-k}t) \end{aligned} \right\} \forall k \in \mathbb{Z}$$

Both T and D are **unitary operators**, i.e., $T^* = T^{-1}$ and $D^* = D^{-1}$. It follows that

$$DT = T^2D \quad \text{and} \quad D^{-1}T = T^{1/2}D^{-1}$$

Scaling Functions and Average Spaces

Definition 1: Multiresolution analysis (MRA) of signals belonging to $L_2(\mathfrak{R})$ is defined as a sequence of closed subspaces V_m , $m \in \mathbb{Z}$ with the following properties:

- $V_{m+1} \subset V_m$
- $h(2t) \in V_m \Leftrightarrow h(t) \in V_{m+1}$
- $h(t) \in V_m \Leftrightarrow h(t+1) \in V_m$
- $\text{Closure}\left(\bigcup_{m=-\infty}^{\infty} V_m\right) = L_2(\mathfrak{R})$ and $\bigcap_{m=-\infty}^{\infty} V_m = \{0\}$
- \exists a **scaling function** $\varphi(\bullet) \in V_0$ such that: *(Explicit expressions for $\varphi(\bullet)$ may not be available)*
 - $\varphi_{m,n}(t) \equiv (D^m T^n \varphi)(t) = \sqrt{2^{-m}} (T^n \varphi)(\sqrt{2^{-m}} t) = \sqrt{2^{-m}} \varphi(2^{-m} t - n)$
 - $\int_{\mathfrak{R}} dt \varphi(t) = 1$, i.e., $\hat{\varphi}(0) = 1$
 - the collection $\{\varphi(t-k) \mid k \in \mathbb{Z}\}$ is an **orthonormal** basis for V_0 ♦

Remark: \exists a sequence $\{h_k\} \in \ell_2(\mathbb{Z})$ such that the **scaling function** satisfies:

$$\varphi(t) = \sqrt{2} \sum_{k \in \mathbb{Z}} h_k \varphi(2t - k) \Rightarrow \sum_{k \in \mathbb{Z}} h_k = 1, \text{ i.e., } \hat{h}(0) = 1 \text{ and } (D\varphi)(t) = \sum_{k \in \mathbb{Z}} h_k T^k \varphi(t)$$

The set $\left\{ \varphi_{j,k} \mid \varphi_{j,k}(t) = \sqrt{2^{-j}} \varphi(2^{-j} t - k); k \in \mathbb{Z} \right\}$ is an **orthonormal** basis for V_j ♦

Wavelet Functions and Detail Spaces

Definition 2: We construct the detail space W_j as orthogonal complement of V_j in V_{j-1} , i.e.,

$$W_j \equiv \{f \in V_{j-1} \subseteq L_2(\mathfrak{R}) : \langle f, g \rangle = 0 \quad \forall g \in V_j\} \Rightarrow V_{j-1} = W_j \oplus V_j$$

The space $L_2(\mathfrak{R})$ can be partitioned as: $L_2(\mathfrak{R}) = V_K \oplus \left(\bigoplus_{j=-\infty}^K W_j \right) \Rightarrow L_2(\mathfrak{R}) = \bigoplus_{j=-\infty}^{\infty} W_j$ ♦

Theorem 1: For any odd integer ℓ , let $g(T) = -T^\ell h^*(-T)$. Then, every $u \in P$, the space of polynomials, can be uniquely expressed as:

$$u(T) = h(T)v(T^2) + g(T)w(T^2) \quad \text{for some } v, w \in P$$

and $W_1 = \text{closure of } \{g(T)w(T^2)\varphi : w \in P\}$ ♦

Definition 3: The mother wavelet $\psi \in W_0 = D^{-1}W_1$ is defined as: $\psi \equiv D^{-1}g(T)\varphi$, i.e.,

$$\psi(t) = \sqrt{2} \sum_{k \in \mathbb{Z}} g_k \varphi(2t - k) \quad \text{so that } W_0 = \text{closure of } \{w(T)\psi : w \in P\}$$

$$\Psi_{m,n} \equiv D^m T^n \psi, \text{ i.e., } \Psi_{m,n}(t) = \sqrt{2^{-m}} \psi(2^{-m}t - n), \text{ that span } W_m$$
 ♦

Theorem 2: The wavelets $\{\Psi_{m,n} : m, n \in \mathbb{Z}\}$ form an orthonormal basis for $L_2(\mathfrak{R})$ ♦

AN EXAMPLE OF SCALING AND WAVELET FUNCTIONS

Haar Scaling Function: $\varphi(t) = \chi_{[0,1)}(t) \equiv \begin{cases} 1 & \forall t \in [0,1) \\ 0 & \forall t \notin [0,1) \end{cases} \Rightarrow \hat{\varphi}(\xi) = e^{-i\pi\xi} \left(\frac{\sin(\pi\xi)}{\pi\xi} \right)$

The projection of a signal $f \in L_2(\mathfrak{R})$ to V_m is:

$$(P_m f)(t) = \left(\sum_{n \in \mathbb{Z}} \varphi_{m,n} \varphi_{m,n}^* \right) f(t) = f_{m,n} \chi_{[2^{-m}n, 2^{-m}(n+1))}(t)$$

where $f_{m,n} \equiv \sqrt{2^{-m}} \varphi_{m,n}^* f = 2^{-m} \int_{2^{-m}n}^{2^{-m}(n+1)} d\theta f(\theta)$ average of f in $[2^{-m}n, 2^{-m}(n+1))$.

The dilation equation is: $D\varphi = \frac{1}{\sqrt{2}} \chi_{[0,2)} = \frac{1}{\sqrt{2}} (\chi_{[0,1)} + \chi_{[1,2)}) = \frac{1}{\sqrt{2}} (I + T)\varphi \Rightarrow h(T) = \frac{1}{\sqrt{2}} (I + T)$

Haar Wavelet Function: $g(T) = -T^\ell h^*(-T) = \frac{1}{\sqrt{2}} (T^{\ell-1} - T^\ell)$ for any odd integer ℓ (by Theorem 1)

Hence, $\psi = D^{-1} g(T) \varphi = \frac{1}{\sqrt{2}} D^{-1} (\chi_{[\ell-1, \ell)} - \chi_{[\ell, \ell+1)}) = \frac{1}{\sqrt{2}} \left(\chi_{[\frac{\ell-1}{2}, \frac{\ell}{2})} - \chi_{[\frac{\ell}{2}, \frac{\ell+1}{2})} \right)$

Setting $\ell = 1$ yields $g(T) = \frac{1}{\sqrt{2}} (I - T)$ and $\psi = \frac{1}{\sqrt{2}} \left(\chi_{[0, \frac{1}{2})} - \chi_{[\frac{1}{2}, 1)} \right)$

Theorem 3 (Janssen): Let φ be a scaling function for an orthogonal MRA. Except for the Haar wavelet, φ cannot be a probability density. ◆

Remark: It is possible to construct biorthogonal bases whose scaling functions are densities. ◆

Implementation of MRA as Filter Banks

Binary Downsampling: $(2_{\downarrow})\{\dots, f_{-2}, f_{-1}, f_0, f_1, f_2, \dots\} \equiv \{\dots, f_{-2}, f_0, f_2, \dots\}$

Binary Upsampling: $(2_{\uparrow})\{\dots, f_{-2}, f_{-1}, f_0, f_1, f_2, \dots\} \equiv \{\dots, 0, f_{-2}, 0, f_{-1}, 0, f_0, 0, f_1, 0, f_2, 0, \dots\}$

Theorem 3: The binary downsampling and upsampling operators have the following properties:

- $2_{\downarrow} = 2_{\uparrow}^*$
- $2_{\downarrow} 2_{\uparrow} = I_{V_0} \Rightarrow (2_{\downarrow} 2_{\uparrow})f = f$ on V_0
- $(2_{\uparrow} u(T))\varphi = u(T^2)\varphi = \sum_{n \in \mathbb{Z}} u_n \varphi_{0,2n}$
- $(2_{\downarrow} v(T))\varphi = \sum_{n \in \mathbb{Z}} v_{2n} \varphi_{0,n}$ and $(2_{\downarrow} T^{-1}v(T))\varphi = \sum_{n \in \mathbb{Z}} v_{2n+1} \varphi_{0,n}$
- $(2_{\downarrow} v(T)2_{\uparrow})\varphi = \sum_{n \in \mathbb{Z}} v_{2n} \varphi_{0,n}$ and $(2_{\downarrow} T^{-1}v(T)2_{\uparrow})\varphi = \sum_{n \in \mathbb{Z}} v_{2n+1} \varphi_{0,n}$
- $2_{\uparrow} 2_{\downarrow} + T 2_{\uparrow} 2_{\downarrow} T^{-1} = I_{V_0}$

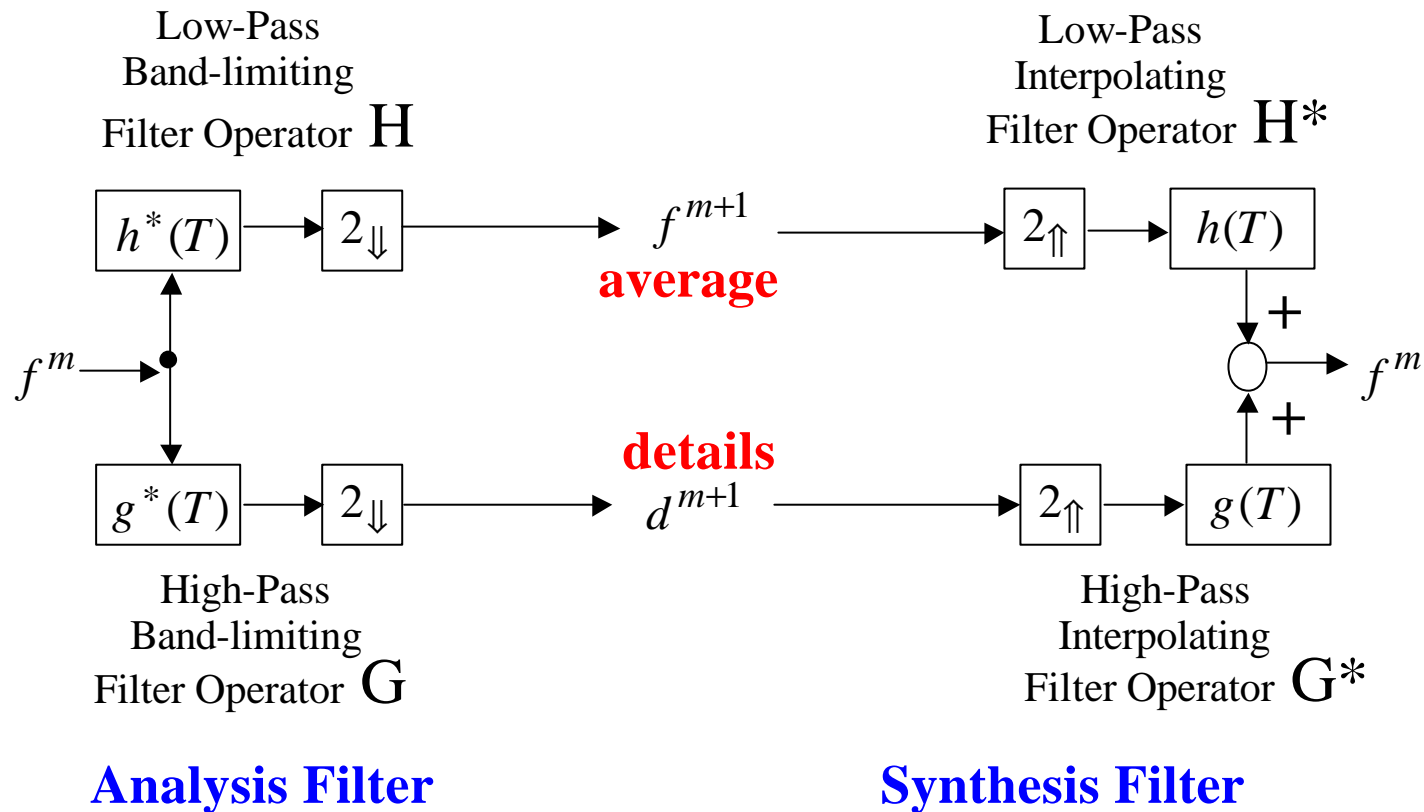


Signal decomposition into 2 frequency bands: low (averaging) and high (details)

Low-pass filter $H^* : V_0 \rightarrow V_0$ defined as: $H^* = h(T)2_{\uparrow} \Leftrightarrow H \equiv 2_{\downarrow} h^*(T)$; and $H^* V_0 = D V_0 = V_1$

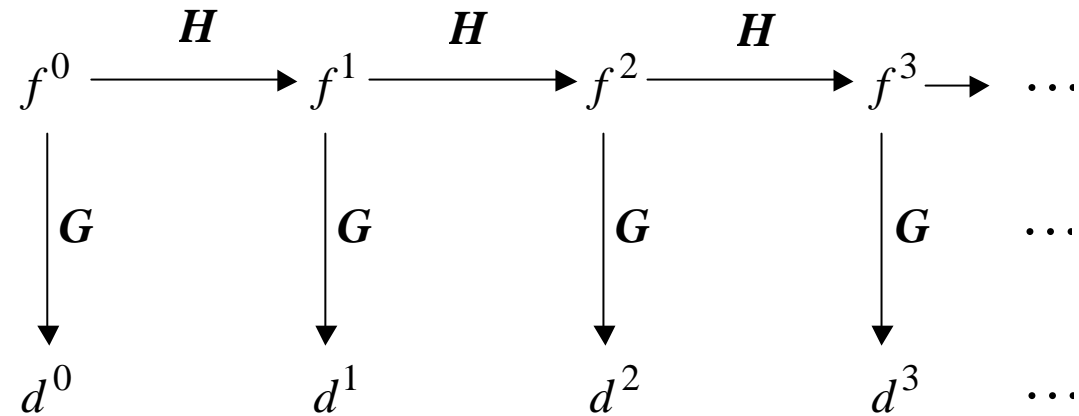
High-pass filter $G^* : W_0 \rightarrow V_0$ defined as: $G^* w(T)\psi \equiv g(T)2_{\uparrow} \varphi \Leftrightarrow G u(T)\varphi \equiv 2_{\downarrow} g^*(T)u(T)\psi$;
and $G^* W_0 = D W_0 = W_1$

Subband Filtering Scheme



Signal Analysis and Synthesis Using Quadrature Mirror Filters (QMF)

Wavelet Decomposition with QMFs



Wavelet Composition with QMFs

