Nondimensionalization of the Navier-Stokes Equation
(Section 10-2, Çengel and Cimbala)

**Nondimensionalization:**
We begin with the differential equation for conservation of linear momentum for a Newtonian fluid, i.e., the **Navier-Stokes equation**. For incompressible flow,

\[
\rho \frac{D\vec{V}}{Dt} = \rho \left[ \frac{\partial \vec{V}}{\partial t} + \left( \vec{V} \cdot \nabla \right) \vec{V} \right] = -\nabla P + \rho \vec{g} + \mu \nabla^2 \vec{V} \tag{10-2}
\]

Equation 10-2 is **dimensional**, and each variable or property (\(\rho, \vec{V}, t, \mu, \text{etc.}\)) is also **dimensional**. What are the primary dimensions (in terms of \{m\}, \{L\}, \{t\}, \{T\}, etc) of each term in this equation?

Answer: \{ \}

To nondimensionalize Eq. 10-2, we choose **scaling parameters** as follows:

<table>
<thead>
<tr>
<th>Scaling Parameter</th>
<th>Description</th>
<th>Primary Dimensions</th>
</tr>
</thead>
<tbody>
<tr>
<td>(L)</td>
<td>Characteristic length</td>
<td>{L}</td>
</tr>
<tr>
<td>(V)</td>
<td>Characteristic speed</td>
<td>{Lt^{-1}}</td>
</tr>
<tr>
<td>(f)</td>
<td>Characteristic frequency</td>
<td>{t^{-1}}</td>
</tr>
<tr>
<td>(P_0 - P_\infty)</td>
<td>Reference pressure difference</td>
<td>{mL^{-1}t^{-2}}</td>
</tr>
<tr>
<td>(g)</td>
<td>Gravitational acceleration</td>
<td>{Lt^{-2}}</td>
</tr>
</tbody>
</table>

We define **nondimensional variables**, using the scaling parameters in Table 10-1:

\[
\begin{align*}
t^* &= ft \\
\vec{x}^* &= \frac{x}{L} \\
\vec{V}^* &= \frac{\vec{V}}{V} \\
P^* &= \frac{P - P_\infty}{P_0 - P_\infty} \\
\vec{g}^* &= \frac{g}{\bar{g}} \\
\vec{V}^* &= L \vec{V}^* \\
\end{align*} \tag{10-3}
\]

To plug Eqs. 10-3 into Eq. 10-2, we need to first rearrange the equations in terms of the dimensional variables, i.e.,

\[
\begin{align*}
t &= \frac{1}{f} t^* \\
\vec{x} &= L \vec{x}^* \\
\vec{V} &= V \vec{V}^* \\
P &= P_\infty + \left( P_0 - P_\infty \right) P^* \\
\vec{g} &= \bar{g} \vec{g}^* \\
\vec{V} &= \frac{1}{L} \vec{V}^* \\
\end{align*}
\]
Now we substitute all of the above into Eq. 10-2 to obtain
\[ \rho VF \frac{\partial \vec{V}^*}{\partial t^*} + \frac{\rho V^2}{L} \left( \vec{V}^* \cdot \vec{V}^* \right) \vec{V}^* = -\frac{P_0 - P_\infty}{\rho V^2} \vec{V}^* P^* + \rho \vec{g}^* + \frac{\mu V}{L^2} \nabla^2 \vec{V}^* \]

Every additive term in the above equation has primary dimensions \( \{m^1 L^{-2} t^{-2}\} \). To nondimensionalize the equation, we multiply every term by constant \( L/(\rho V^2) \), which has primary dimensions \( \{m^{-1} L^2 t^2\} \), so that the dimensions cancel. After some rearrangement,

\[ \left[ \frac{fL}{V} \right] \frac{\partial \vec{V}^*}{\partial t^*} + \left( \vec{V}^* \cdot \vec{V}^* \right) \vec{V}^* = -\left[ \frac{P_0 - P_\infty}{\rho V^2} \right] \vec{V}^* P^* + \left[ \frac{gL^2}{V^2} \right] \vec{g}^* + \left[ \frac{\mu}{\rho VL} \right] \nabla^2 \vec{V}^* \] \hspace{1cm} (10–5)

Thus, Eq. 10-5 can therefore be written as

\[ [St] \frac{\partial \vec{V}^*}{\partial t^*} + (\vec{V}^* \cdot \vec{V}^*) \vec{V}^* = -[Eu] \vec{V}^* P^* + \left[ \frac{1}{Fr^2} \right] \vec{g}^* + \left[ \frac{1}{Re} \right] \nabla^2 \vec{V}^* \] \hspace{1cm} (10–6)

**Nondimensionalization vs. Normalization:**

Equation 10-6 above is *nondimensional*, but not necessarily *normalized*. What is the difference?

- **Nondimensionalization** concerns only the *dimensions* of the equation – we can use *any* value of scaling parameters \( L, V \), etc., and we always end up with Eq. 10-6.
- **Normalization** is more restrictive than nondimensionalization. To *normalize* the equation, we must choose scaling parameters \( L, V \), etc. that are appropriate for the flow being analyzed, such that all nondimensional variables \((t^*, \vec{V}^*, P^*, \text{etc.})\) in Eq. 10-6 are of order of magnitude unity. In other words, their minimum and maximum values are reasonably close to 1.0 (e.g., \(-6 < P^* < 3\), or \(0 < P^* < 11\), but *not* \(0 < P^* < 0.001\), or \(-200 < P^* < 500\)). We express the normalization as follows:

\[ t^* \sim 1, \quad \bar{x}^* \sim 1, \quad \vec{V}^* \sim 1, \quad P^* \sim 1, \quad \vec{g}^* \sim 1, \quad \nabla^2 \vec{V}^* \sim 1 \]

If we have properly normalized the Navier-Stokes equation, we can compare the relative importance of various terms in the equation by comparing the *relative magnitudes* of the nondimensional parameters \( St, Eu, Fr, \text{and} Re \).