Chapter 01: Dynamical Systems

Source

- Thermodynamics of Chaotic systems by C. Beck and F Schlögl (1993)
- Lectures on Geometry and Dynamical Systems by Y. Pesin and V. Clemenhaga (2009)
- Chaos in Dynamical Systems by E. Ott (2002)

Topics

- Continuous-time dynamical systems
- Discrete-time dynamical systems
- Fixed points, periodic points, and other attractors
- Stability of fixed points and periodic points with examples
- Topological conjugation
- Bifurcation with examples
- Concept of Chaos
- Analysis of chaotic systems and symbolic dynamics
A dynamical system is represented by a deterministic mathematical model that describes the state of a system evolving in forward time, where time could be either discrete or continuous. The Appendix provides a formal introduction to dynamical systems.

1.1 Dynamical systems in discrete time

Let $\mathbb{R}^n$ be the $n$-dimensional Euclidean space, where $n$ is a positive integer. This space is the collection of $n$-element vectors, each of which is an $n$-tuple of real numbers. As a dynamical system $f$, we may consider any map that takes one element of $\mathbb{R}^n$ and returns another element of $\mathbb{R}^n$. Furthermore, the map $f$ could be defined on a subset $D \subseteq \mathbb{R}^n$, i.e., $D$ is the domain of the map $f$ and $f(x)$ is the image of $x \in D$; then, $x$ is a preimage (not necessarily inverse) of $f(x)$. It should be noted that, given an $x$, its image should be unique. However, a single point $f(x)$ in range space of $f$ may have multiple preimages. Let us consider a set $A \subseteq D$. The image of the set $A$ is denoted as the set

$$f(A) = \{f(x)|x \in A\}$$

(1)

The examples of interest are those, where $f(D) \subseteq D$; that is, the image $f(D)$ is contained within $D$. Such maps could be used recursively.

A subset $A \subseteq D$, for which $f(A) \subseteq A$ is said to be invariant under the map $f$. In other words, if the range of $f$ is $D$, then the map $f$ can be applied recursively without leaving $D$. Thus, it is logical to consider not only the map $f$ itself, but also the map obtained by applying $f$ twice, denoted as $f^2(x) = (f \circ f)(x) = f(f(x))$. In general, $n$ iterations of $f$ is written as:

$$f^n = f \circ f \circ \ldots \circ f \quad (n \text{ times})$$

(2)

Naturally, $f^{n+m} = f^n \circ f^m = f^m \circ f^n$

Trajectory: The trajectory of a point $x$, also called an orbit, is the sequence of points $x, f(x), f^2(x), \ldots$. This sequence tracks the evolution of the state $x$ upon repeated actions of the map $f$.

Preimage: In general, the map $f$ is not required to be bijective (i.e., one-to-one correspondence), therefore, not invertible. In that case, the issue of time running backward is not a straight-forward concept because, given a point
x, it may not be possible to obtain the unique past/history of the point x. However, it is possible to define a set function as stated below.

\[ f^{-1}(A) = \{ x \in D | f(x) \in A \} \]  

(3)

which contains the set of all points whose images lie in A.

**Fixed Points:** A fixed point of a map is invariant to the operations of the map. That is, x is said to be a fixed point if \( f(x) = x \). Maps may also contain periodic orbits that repeat after a certain length/period \( k \), where the integer \( k \geq 1 \). That is, x lies on an orbit with period \( k \) if \( k \) is the smallest possible natural number such that \( f^k(x) = x \).

**Remark 1.1.** A point that lies on an orbit of period \( k \) for the map \( f \), is the fixed point for the map \( f^k \). However a trajectory may stay aperiodic forever. This aspect is explained in detail later.

1.2 Dynamical systems in continuous time

A dynamical system is expressed in the form of ordinary differential equations (ODEs) as:

\[ \frac{d}{dt}x(t) = F(x(t)) \]  

(4)

where \( x : (a, b) \to \Omega \subseteq \mathbb{R}^n \) and \( (a, b) \subseteq \mathbb{R} \).

It is noted that a discrete-time dynamical system may exhibit a chaotic behavior in any dimension (\( \geq 1 \)), while continuous-time autonomous systems cannot be chaotic unless its dimension is three or higher. Note that, for an invertible (discrete-time) map, a chaotic behavior exists only if the state space dimension is greater than or equal to 2.

For a given initial condition \( x(0) \), the solution to the differential equation (4) may lead to a trajectory \( x(t) \). Such a trajectory exists and is unique if the Lipschitz condition is satisfied, i.e., there exists positive real \( M \in (0, \infty) \) such that

\[ \| F(x) - F(y) \| \leq M \| x - y \| \quad \forall x, y \in \Omega \]  

(5)

where \( \| \Theta \| \) is a norm of \( \Theta \in \mathbb{R}^n \).

If the function \( F \) in Eq. (4) is continuously differentiable, i.e., if all the partial derivatives \( \frac{\partial F_i}{\partial x_j} \) exist and are continuous, then the standard existence and
uniqueness theorem from the theory of ODEs implies that the system has a unique solution on small intervals \([\tau - \epsilon, \tau + \epsilon]\), i.e., \(\tau - \epsilon \leq t \leq \tau + \epsilon\). By gluing together the solutions on these intervals, the solution may be extended over the maximal interval \(t_0 \leq t < t_f\); the endpoints \(t_0\) and \(t_f\) are allowed to be either infinite or the points at which \(x(t)\) reaches the boundary of \(\Omega\). If \(\Omega\) is unbounded, then it is possible for a solution \(x(t)\) to reach infinity in finite time. This phenomenon is called finite-time escape.

A family of maps \(\phi_t : \Omega \rightarrow \Omega\), parameterized by \(t\), is defined such that \(\phi_t(x(0)) = x(t)\). Unlike discrete-time systems, it is possible to go back in time for continuous systems. Therefore, the Abelian group property holds:

\[ \phi_{t+s} = \phi_t \circ \phi_s = \phi_s \circ \phi_t \]  

(6)

The simple approach to creating a map from a flow is to sample the flow at discrete times \(t_n = t_0 + nT\), where \(T\) is the (uniform) sampling time.

Another technique for reduction of a continuous-time dynamical system to a discrete-time is called the Poincaré section. Let \(S\) be an \((N-1)\)-dimensional Poincaré section of an \(N\)-dimensional continuous-time system. The Poincaré map \(P : S \rightarrow S\) maps a point \(x \in S\) to the point where the flow emanating from \(x\) first intersects the surface \(S\). The behaviors of the Poincaré map and the original dynamical system is similar in terms of stability properties.

1.3 Attractors

An attractor is a subset \(A\) of the phase space characterized by the following three conditions:

1. \(A\) is forward invariant under a mapping \(f\) if \(f(a)\) is an element of \(A\) for all \(a \in A\) and for all \(t > 0\).

2. There exists a neighborhood of \(A\), called the basin of attraction for \(A\), denoted as \(B(A)\), which consists of all points \(b\) that enter \(A\) in the limit \(t \rightarrow \infty\). More formally, \(B(A)\) is the set of all points \(b\) in the phase space with the following property:

   For any open neighborhood \(N\) of \(A\), there is a positive constant \(T\) such that \(f_t(b) \in N\) for all \(t > T\).

3. There is no proper subset of \(A\), which satisfies the first two properties.
Geometrically, an attractor can be a point, a curve, a manifold, or even a complicated set with a fractal structure known as a strange attractor. Types of attractors include the following:

- **Fixed point**: A fixed point is a point of a function that does not change under the transformation, i.e., for a map $f : S \rightarrow S$, a point $x \in S$ is a fixed point of $f$ if $f(x) = x$.

- **Limit cycle**: A limit cycle is a periodic orbit of the system that is isolated.

- **Limit tori**: There may be more than one frequency in the periodic trajectory of the system through the state of a limit cycle. If two of these frequencies form an irrational fraction (i.e. they are incommensurate), the trajectory is no longer closed, and the limit cycle becomes a limit torus.

- **Strange Attractor**: An attractor is informally described as strange if it has non-integer dimension. This is often the case when the dynamics on it are chaotic, but there exist also strange attractors that are not chaotic; the converse is also true.

1.4 Sensitivity to initial conditions

A defining attribute of an attractor on which the dynamics are chaotic is that it displays exponential (local) sensitivity to initial conditions. Let $x_1(0)$ and $x_2(0) = x_1(0) + \Delta(0)$ be two initial conditions of a dynamical system and let the corresponding orbits be denoted by $x_1(t)$ and $x_2(t)$. Let the separation between these two orbits be denoted as: $\Delta(t) \triangleq x_2(t) - x_1(t)$. If, for $\Delta(0) \rightarrow 0$ and a sufficiently large $t$, the orbit remains bounded and $|\Delta(t)| \sim |\Delta(0)| \exp(ht)$, $h > 0$, i.e., $|\Delta(t)|$ grows exponentially, then the system displays sensitive dependence on the initial condition. The implication is that small errors in the initial condition may eventually grow rapidly with time. Examples are computer round-off errors and noise that can totally change the trajectory after a sufficiently long time.

1.5 Stability - Fixed points & periodic orbits

This section presents the analysis of two types of attractors: fixed points and periodic orbits. Let $p_0$ be a fixed point, i.e., $f(p_0) = p_0$, for a map $f$ operating
on $\mathbb{R}^n \rightarrow \mathbb{R}^n$. If $f$ is continuously differentiable in the neighborhood of $p_0$, then it follows that

$$f(p) = f(p_0) + Df(p_0)(p - p_0) + o(p - p_0)$$  \hspace{1cm} (7)

where $Df(p_0)$ is the local partial derivative matrix, called Jacobian, at $p_0$ that is expressed as:

$$Df(p_0) = \begin{bmatrix}
\frac{df_1}{dx_1} & \cdots & \frac{df_1}{dx_n} \\
\vdots & \ddots & \vdots \\
\frac{df_n}{dx_1} & \cdots & \frac{df_n}{dx_n}
\end{bmatrix}_{|p_0}$$  \hspace{1cm} (8)

**Theorem 1.1** (Hartman-Grobman Theorem). If all eigenvalues $\lambda$ of $Df(p_0)$ are confined within the unit disk, i.e., if there exists an $\alpha < 1$ such that, $|\lambda| < \alpha$, then there exists an $\epsilon > 0$ such that, for all $p \in \mathbb{R}^n$ with $\|p - p_0\| < \epsilon$, the limit $\lim_{k \to \infty} f^k(p) = p_0$. Note that $\| \cdot \|$ denotes a norm and all norms are equivalent on a finite-dimensional vector space.

**Remark 1.2.** A consequence of Hartman-Grobman theorem is that if all the eigenvalues of the Jacobian of a fixed point lie within the unit disk, then the fixed point is locally stable.

The case, where some of the eigenvalues of the Jacobian lie within the unit disk and some lie outside the unit disk, is harder to analyze, because one must first prove the existence of stable and unstable manifolds. These are curves/surfaces $\gamma^u$ and $\gamma^s$ passing through $p_0$ such that:

- $\gamma^s$ and $\gamma^u$ are $f$-invariant, i.e., if $x \in \gamma^s$, then $f(x) \in \gamma^s$ and similarly, if $x \in \gamma^u$, then $f(x) \in \gamma^u$.
- The tangent vector to $\gamma^s$ at $p_0$ points in the direction of the eigenvectors corresponding to the eigenvalues inside the unit circle. Similarly, the tangent to $\gamma^u$ corresponds to the other eigenvectors.
- Trajectories on $\gamma^s$ are attracted toward and eventually reach $p_0$, while trajectories on $\gamma^u$ are repelled away from $p_0$.

Let us consider an example of a second-order autonomous system, where $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ that has a fixed point at $p_0 \in \mathbb{R}^2$; let $\lambda$ and $\mu$ be the eigenvalues of $Df(p_0)$, with $|\lambda| \leq |\mu|$. Assuming that neither $\lambda$ nor $\mu$ lies on the boundary of the unit disk (i.e., the unit circle $S^1$), there are three distinct characteristics of the trajectories in the neighborhood of the fixed point $p_0$. 


1. (Attracting behavior) If $|\lambda| \leq |\mu| < 1$, then all trajectories converge to $p_0$; $p_0$ is an attracting fixed point. If the eigenvalues form a complex conjugate pair, then the trajectories move along a logarithmic spiral as seen in Fig. 1(a)). If both $\lambda$ and $\mu$ are real, then trajectories move along the paths in Fig. 1(b) if $Df(p_0)$ is diagonalizable, or along the paths in Fig. 1(c) if the Jacobian matrix $Df(p_0)$ is similar to $\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$. Note that, for the trajectories in Fig. 1(c) to occur, there must be repeated eigenvalues, i.e., $\lambda = \mu$.

![Figure 1: Trajectories near attracting fixed points](image)

2. (Hyperbolic behavior) If $|\lambda| < 1 < |\mu|$, the the fixed point $p_0$ is called hyperbolic, also called a saddle, as seen in Fig. 2. From the stable direction (i.e., abscissa in Fig. 2) that corresponds to the eigen-direction for the stable eigenvalue $\lambda$, trajectories approach $p_0$ as $k \to \infty$. From the unstable direction (i.e., the ordinate in Fig. 2), corresponding to the eigen-direction for the unstable eigenvalue $\mu$, the trajectories move away from $p_0$ as $k \to \infty$. All other trajectories follow hyperbola-like paths, at first moving closer to $p_0$, and then moving away from $p_0$.

3. (Repelling behavior) $1 < |\lambda| \leq |\mu|$: All trajectories move away from the fixed point $p_0$; so $p_0$ is a repelling fixed point. Trajectories move along one of the curves similar to those in in Fig. 1, but in the opposite direction.

Stability of periodic points can be analyzed in a similar way. Let us assume that there exists a periodic orbit (with period $= m$) through points $p_0, p_1, \ldots, p_{m-1}$. That is, $f(p_0) = p_1, f(p_1) = p_2, \ldots, f(p_{m-1}) = p_0$. Naturally, each of these points would be the fixed points for the map $f^m$. Therefore, stability of the orbit could be determined from the eigenvalues of the
Jacobian $Df^k(p_0)$. It is noted that $Df^k(p_0) = Df^k(p_1) = \ldots = Df^k(p_{m-1})$, and

$$Df^k(p_0) = \prod_{i} Df(p_i)$$  \hspace{1cm} (9)

### 1.6 An Example - Logistic Map

Let a map $f$ be defined on the domain and range $[0,1]$, i.e., $f : [0,1] \rightarrow [0,1]$. 

$$f(x) = rx(1-x), \quad r > 0$$  \hspace{1cm} (10)

This map could be viewed as a simple ecological model (e.g., population growth rate in species), where the growth rate is proportional to the current population. However, due to limited resources (e.g., food), the growth rate falls if the population becomes too large. Figure 3 shows the trajectory for $r = 2.8$ and $x_0 = 0.1$. It follows from the above equation that $Df(x) = r(1-2x)$. Then, for $r = 2.8$, the map has two fixed points: one at 0 and the other at $\approx 0.6429$. Upon linearizing around $x = 0$, $Df(0) = 2.8$ (unstable) and around $x = \approx 0.6429$, $Df(0.6429) = -0.8$ (stable).

Figure 4 shows the behavior of the logistic map for $r = 3.15$. In this case too, the map has two fixed points at 0 and $\approx 0.6825$. However, both fixed points are unstable because $Df(0) = 3.15$ and $Df(0.6825) = -1.1498$. A periodic orbit with period 2 is observed because a periodic orbit with period 2 is a fixed point of $f^2$. This periodic orbit passes through points $\approx 0.7840$ and $\approx 0.5335$. Since $Df(0.7840) = Df(0.5335) = 0.1936$, this periodic orbit is stable.
2 Topological Conjugation

Now we introduce the concept of topological conjugation. The notion of conjugacy is related to existence of an invertible map that links two dynamical systems; and topological conjugacy results from a combination of conjugacy and homeomorphism (i.e., a bijectivity and bicontinuity). A formal definition follows.

**Definition 2.1** (Topological Conjugacy). Let $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^n$; let $f : X \to X$ and $g : Y \to Y$ be two maps. Then, $f$ and $g$ are defined to be topologically conjugate if there exists a homeomorphism $\phi : Y \to X$ such that $f \circ \phi = \phi \circ g$.

**Remark 2.1.** In the above definition, $f \circ \phi = \phi \circ g$ implies that $f(\phi(y)) = \phi(g(y))$. Thus, $g = \phi^{-1} \circ f \circ \phi$ and $f = \phi \circ g \circ \phi^{-1}$ because of bijectivity of $\phi$. Furthermore, since $\phi$ is bicontinuous, $g(\phi^{-1}(x)) = \phi^{-1}(f(x))$. Topological
conjugation could be thought of as a change of coordinates, where the conjugating homeomorphism $\phi$ is the function that transforms the coordinates. Two topologically conjugate maps have the same topological properties that include the qualitative behavior in terms of fixed points, period orbits and other attractors and their stability.

**Example 2.1.** This example deals with parameterized families of the Ulam map $f_\mu : X \to X$ and the logistic map $g_r(y) : Y \to Y$. Let $f_\mu(x) = 1 - \mu x^2$ and $g_r(y) = ry(1 - y)$, where $\mu$ and $r$ are scalar parameters. By setting an affine relation in the form $\phi(y) = ay + b$, where $a$ and $b$ are real constants, it follows from the condition, $g = \phi^{-1} \circ f \circ \phi$, of topological conjugacy and equality of like powers of $y$ that

$$\phi(y) = \left(y - \frac{1}{2}\right) \frac{1}{\mu} \left((1 + 4\mu)^{\frac{1}{2}} + 1\right) \text{ and } r = (1 + 4\mu)^{\frac{1}{2}} + 1 \quad (11)$$

Setting the parameter $\mu = 2$ and the space $X = [-1, 1]$, the Ulam map becomes $f(x) = 1 - 2x^2$. Similarly, setting the parameter $r = 4$ and the space $Y = [0, 1]$, the logistic map becomes $g(y) = 4y(1 - y)$. Since $\phi$ is an affine function, it is both bijective and bicontinuous, it is a homeomorphism.

**Example 2.2.** We consider another example to construct a topological conjugacy between the Ulam map and the tent map $g : [0, 1] \to [0, 1]$ that is defined as:

$$g(y) = \begin{cases} 2y & \text{if } y \in [0, 0.5) \\ 2(1 - y) & \text{if } y \in [0.5, 1.0] \end{cases}$$

The affine function $\phi : [-1, 1] \to [0, 1]$ in Example 2.1 is replaced by a smooth nonlinear function $\phi(y) = -\cos(\pi y) = x$ in this example. Note that the space $X = [-1, 1]$ and $Y = [0, 1]$, and the parameter $\mu = 2$ remain the same as in Example 2.1. It is also evident that $\phi : [-1, 1] \to [0, 1]$ is a homeomorphism. It follows from the condition of topological conjugacy that

$$g(y) = \phi^{-1}(f(\phi(y))) \quad (12)$$

$$g(y) = \frac{-1}{\pi} \cos^{-1} \left(1 - 2\cos^2(\pi y)\right) \quad (13)$$

$$g(y) = \frac{-1}{\pi} \cos^{-1} \left(1 - 2\cos^2(\pi y)\right) \quad (14)$$
\begin{equation}
    g(y) = \frac{-1}{\pi} \cos^{-1}(-\cos(2\pi y))
\end{equation}

\begin{equation}
    g(y) = 2y, \quad y \in [0, 0.5] \quad \text{and} \quad g(y) = 2(y - 1), \quad y \in [0.5, 1]
\end{equation}

2.1 Bifurcation

This subsection introduces the concept of bifurcation in nonlinear dynamical systems. A general notion of bifurcation is a qualitative change in the dynamical behavior as a system parameter varies. Here, let us focus on bifurcations of smooth maps that depend on a single parameter (e.g., the parameter \( r \) in the (one-dimensional) logistic map \( x_{k+1} = rx_k(1-x_k) \)). Note that a function is "smooth" if it is continuous and several times differentiable for all values of its argument.

**Definition 2.2** (Bifurcation). A single-parameter family of maps \( f_c : X \to X \) is defined to have a bifurcation at \( c_0 \) if, for all \( \epsilon > 0 \), there exists a parameter value \( c \in (c_0 - \epsilon, c_0 + \epsilon) \) for which \( f_c \) and \( f_{c_0} \) are not topologically conjugate.

Definition 2.2 implies that the qualitative behavior of the dynamical system (the attractors and their stability) are different for parameter values greater than and less that \( c \). Often a very small change in the value of a system parameter could lead to a very change in dynamical behavior of the system.

The concept of bifurcation in nonlinear dynamics is analogous to that of phase transition in statistical mechanics. Let us elucidate a few common types of bifurcations. Note that this is not an exhaustive list of bifurcations.

**Definition 2.3** (Saddle-node (tangent) bifurcation). A single-parameter family of maps \( f_c : X \to X \) is defined to have a saddle-node (or tangent) bifurcation at \( c_0 \) if there exists an open set \( I \subseteq X \) and a real number \( \epsilon > 0 \) such that the following conditions hold:

1. For \( c_0 - \epsilon < c < c_0 \), the map \( f_c \) has no fixed points in \( I \).

2. For \( c_0 = c \), the map \( f_c \) has one fixed point in \( I \), which is neutral (i.e., the Jacobian has a unit eigenvalue).

3. For \( c_0 < c < c_0 + \epsilon \), the map \( f_c \) has two fixed points in \( I \), of which one is attracting and the other is repelling.
Definition 2.3 implies that, upon perturbation of the parameter $c$, two new fixed points could be created – one being stable and the other is unstable. Figure 5 illustrates the concept.

![Figure 5: Saddle-node (tangent) bifurcation](image)

**Definition 2.4** (Period-doubling (pitchfork) bifurcation). A single-parameter family of maps $f_c : X \to X$ is defined to have a period-doubling (or pitchfork) bifurcation at $c_0$ if there exists an open set $I \subseteq X$ and a real number $\epsilon > 0$ such that the following conditions hold:

1. For all $c_0 - \epsilon < c < c_0 + \epsilon$, the map $f_c$ has a unique fixed point $p_c$ in $I$.
2. For $c_0 - \epsilon < c < c_0$, the fixed point $p_c$ is attracting, and $f_c$ has no period-two cycle in $I$.
3. For $c_0 < c < c_0 + \epsilon$, the fixed point $p_c$ is repelling, and there exists a unique period-two cycle $(q^1_c, q^2_c)$ in $I$, which is attracting.
4. As $c$ converges to $c_0$, both points $q^i_c$ of the period-two cycle converge to the fixed point $p_c$.

Definition 2.4 implies that the stable fixed point becomes unstable while giving rise to a stable period-two orbit. The converse is also possible, where an unstable fixed point becomes stable and an unstable period-two orbit is created. Figure 6 illustrates the concept. Note that this explanation only addresses generation of a period-two orbit from a fixed point. In general, such a process is similar when the period changes from 2 to 4 and then to 8, 16, etc. The bifurcation that changes the period from 2 to 4 is often called a pitchfork bifurcation in $f^2_c$. 

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3 Introduction to chaos

Dynamical systems, represented by linear (finite-dimensional time-invariant) models, are usually simple to understand and their behavior is tractable. The chaotic behavior, observed in dynamical systems (e.g., the Logistic map and the Henon map) are attributed to their nonlinearities; chaos may occur even if a nonlinear dynamical system map is of very low order. Let us analyze the chaotic behavior of the following piecewise-linear map.

\[ f(x) = \begin{cases} 
3x & x \in [0, 1/3] \\
3x - 2 & x \in [2/3, 1]
\end{cases} \]

(17)

The map is defined over two intervals \( I_0 = [0, 1/3] \) and \( I_2 = [2/3, 1] \). Notice that images \( f(I_0) = f(I_2) = [0, 1] \). However, since the domain and range of \( f \) are not the same, \( f \) is not qualified to be a dynamical system on \( I_0 \cup I_2 \). The aim here is to obtain the largest invariant set for \( f \) and subsequently examine the dynamics within the invariant set.

To begin with, any \( x \in D \triangleq I_0 \cup I_2 \) permits one iteration. In order for \( f^2(x) \) to be defined, both \( x \) and \( f(x) \) must lie in the domain of \( f \), i.e., the following condition must be satisfied: \( x \in D \cap f^{-1}(D) \).
The domain of \( f^2 \) is a collection of four intervals: \( I_{00} = [0, 1/9] \), \( I_{02} = [2/9, 3/9] \), \( I_{20} = [2/3, 7/9] \) and \( I_{22} = [8/9, 1] \).

\[
I_{i_1i_2} = I_{i_1} \cap f^{-1}(I_{i_2})
\]  

(18)

The domain of \( f \) that permits three iterations consists of eight (\( 2^3 = 8 \)) intervals \( I_{000}, I_{002}, I_{020}, I_{022}, I_{200}, I_{202}, I_{220} \) and \( I_{222} \). Inductively, the \( 2^n \) intervals in the \( n^{th} \) iteration are given as:

\[
I_{i_1i_2i_3 \ldots i_n} = I_{i_1} \cap f^{-1}(I_{i_2}) \cap f^{-2}(I_{i_3}) \cap \ldots \cap f^{-n+1}(I_{i_n})
\]  

(19)

The domain of the \( f^n \) is given by:

\[
D_n \triangleq \bigcup_{(i_1i_2 \ldots i_n)} I_{i_1i_2i_3 \ldots i_n}
\]  

(20)

and the domain on every iterate of \( f \) is obtained as:

\[
C \triangleq \bigcap_{n \geq 1} D_n = \bigcap_{n \geq 1} \bigcup_{(i_1i_2 \ldots i_n)} I_{i_1i_2i_3 \ldots i_n}
\]  

(21)

where \( i_k \) is either 0 or 2, and \( C \) is called the one-third Cantor set.

Therefore, \( f : C \to C \) is a dynamical system. The properties of sets like \( C \) will be examined later.

The dynamical behavior of \( C \) is sensitive to its initial condition. That is, if \( x,y \in C \) are sufficiently close to each other, then \( d(f^n(x), f^n(y)) = 3^nd(x,y) \), provided that \( f^k(x) \) and \( f^k(y) \) lie in the same interval \( I_k \) for all \( 1 \leq k \leq n \). Therefore, nearby trajectories repel each other at an exponential rate.

Every point \( x \in C \) can be uniquely determined by a sequence if numbers \( i_1, i_2, i_3, \ldots \), each of which is either 0 or 2.

\[
x \in I_{i_1} \cap I_{i_1i_2} \cap I_{i_1i_2i_3} \ldots
\]

(22)

Next let us introduce the concept of a symbolic space defined the symbol alphabet \( \{0, 2\} \).

**Definition 3.1 (Symbolic Space).** The space of all symbolic sequences on the alphabet \( \{1, 2\} \) is defined as:

\[
\Sigma_2^+ = \{0, 2\}^\infty = \{(i_k)_{k=1}^\infty : i_k \in \{0, 2\} \ \forall k\}
\]  

(23)

A (bijective) coding map \( h : \Sigma_2^+ \to C \) given by the following rule:
Remark 3.1. It can be shown that \( h \) is a homeomorphism with an appropriately chosen metric in the symbolic space. Therefore, the dynamics defined on \( C \) and \( \Sigma_2^+ \) are topologically conjugate.

Let us develop the dynamics on the symbolic space. The coding of a symbol sequence \( w \in \Sigma_2^+ \) is given by

\[
h(w) = h(i_1, i_2, i_3, \ldots) = \bigcap_{n=1}^{\infty} I_{i_1 i_2 i_3 \ldots i_n}
\]

(24)

\[
h(w) = \bigcap_{n=1}^{\infty} f^{-(n-1)}(I_{i_n}) = I_{i_1} \cap f^{-1}(I_{i_2}) \cap f^{-2}(I_{i_3}) \cap \ldots
\]

(25)

The map \( f \) imposes the dynamics on \( h(w) \) as

\[
f(h(w)) = f(I_{i_1}) \cap I_{i_2} \cap f^{-1}(I_{i_3}) \cap f^{-2}(I_{i_4}) \cap \ldots
\]

\[
= h(w')
\]

and \( w' \) is expressed as a sequence \( \{i_2, i_3, i_4, \ldots\} \) based on the fact that \( f(I_{i_1}) = [0, 1] \) and \( f(f^{-1}(X)) = X \) for any set \( X \) in the range of \( f \). Therefore, the map that takes \( w \) to \( w' \) just shifts the sequence one position to the left. This is called the shift map \( \sigma \) as defined below.

\[
\sigma : \Sigma_2^+ \rightarrow \Sigma_2^+
\]

\[
\sigma : (i_1, i_2, i_3, \ldots) \mapsto (i_2, i_3, i_4, \ldots)
\]

(27)

Since \( h \) is a homeomorphism, the dynamical systems \( f : C \rightarrow C \) and \( \sigma : \Sigma_2^+ \rightarrow \Sigma_2^+ \) are topologically conjugate. Thus, it suffices to study the dynamics in the symbolic space due to the shift map and extend it to the domain of \( f \). The conjugacy induced by \( h \) may be used to realize the dynamics of \( f \) on \( C \).

As an exercise, see where some these periodic trajectories lie in the domain \( C \).

4 Analysis of Symbolic Sequences

This section describes the map \( \sigma : \Sigma_2^+ \rightarrow \Sigma_2^+ \) in terms of fixed points, periodic points and chaotic orbits.

- **Fixed points**: For a fixed point we must have \( i_n = i_{n+1} \). Thus, \( \Sigma_2^+ \) contains only two fixed points: \( \{0, 0, 0, \ldots\} \) and \( \{2, 2, 2, \ldots\} \). As an exercise, see what these fixed points correspond to the domain \( C \).
Periodic Trajectories: Consider two sequences: \( \{0, 2, 0, 2, 0, 2, \ldots \} \) and \( \{2, 0, 2, 0, 2, 0, \ldots \} \), which satisfy the equation \( \sigma^2(w) = w \) in addition to the fixed points that also satisfy the equation. Both these sequences are period-two orbits. Extending this argument, we may construct period-3 orbits such as \( \{0, 0, 2, 0, 2, 0, 2, 0, \ldots \} \). In general, \( \sigma^m(w) = w \) has \( 2^m \) solutions in \( w \). However, some of them may have a period smaller than \( m \). Another consequence is that it is possible to construct periodic orbits of any arbitrary period and hence the number of periodic orbits is at most countably infinite.

Chotic Trajectories: There exists an uncountable number of symbolic sequences that are neither fixed points nor periodic orbits. These are sequences in \( \Sigma_2^+ \), which are non repeating.

To understand the aspects of cardinality of the set of periodic orbits and chaotic trajectories the following method may prove to be useful.

Let \( w = (i_1, i_2, \ldots) \in \Sigma_2^+ \) and all 2’s in the sequence be replaced by 1. Now, the new sequence only involves 0’s and 1’s, and may be viewed as consisting of the numbers after the decimal point in the binary representation of real numbers in \([0, 1]\). This is analogous to the fact that all rational numbers in \([0, 1]\) produce a recurring binary notation and thus correspond to either periodic orbits or trajectories terminating on periodic orbits. On the other hand, the irrational numbers in \([0, 1]\) produce non-recurring binary numbers and thus correspond to chaotic orbits. This observation is summarized as follows.

*The cardinality of periodic orbits is the same as the cardinality of rational numbers, while the cardinality of chaotic trajectories is the same as that of irrational numbers.*
This appendix presents a brief overview of Systems Theory to introduce the concept of dynamical systems. In literature, the term ”system(s)” is often used very widely and broadly, which tends to connote imprecise concepts. In this appendix, a system, more accurately a dynamical system, is a clearly defined object. This is in agreement with the opinion of many scientists who treat Systems Theory largely, although not entirely, as a branch of mathematics.

Basic Definitions

This subsection presents a few definitions to formally introduce the notion of dynamical systems.

Definition 4.1. A dynamical system $\mathcal{S}$ is a composite mathematical concept defined by the following axioms:

1. There exists a time set $\mathbb{T}$, a state set $\mathbb{X}$, an input value set $\mathbb{U}$, an input excitation space $\Omega = \{\omega : \mathbb{T} \rightarrow \mathbb{U}\}$, an output value set $\mathbb{Y}$, and an output space $\Gamma = \{\gamma : \mathbb{T} \rightarrow \mathbb{Y}\}$.

2. $\mathbb{T}$ is an ordered subset of the real field $\mathbb{R}$.

3. The input space $\Omega$ satisfies the following conditions:
   
   (i) (Nontriviality): $\Omega$ is non-empty.
   
   (ii) (Concatenation of inputs): An input segment $\omega_{[\tau,t]}$ is realized as a restriction of $\omega \in \Omega$ to $(\tau,t] \cap \mathbb{T}$. If $\omega, \bar{\omega} \in \Omega$ and $t_1 < t_2 < t_3$, then there exists an $\omega^* \in \Omega$ such that $\omega_{[t_1,t_2]}^* = \omega_{(t_1,t_2]}$ and $\omega_{[t_2,t_3]}^* = \bar{\omega}_{(t_2,t_3]}$.

4. There exists a state transition function $\varphi : \mathbb{T} \times \mathbb{T} \times \mathbb{X} \times \Omega \rightarrow \mathbb{X}$ where the state $x(t) = \varphi(t; \tau, x(\tau), \omega)$ $\in \mathbb{X}$ resulting at time $t \in \mathbb{T}$ from the initial state $x(\tau)$ at the initial time $\tau \in \mathbb{T}$ under the excitation of the input $\omega \in \Omega$. Furthermore, $\varphi$ has the following properties:
   
   (i) (Direction of time): $\varphi(t; \tau, x(\tau), \omega)$ is defined for all $t \geq \tau$, but not necessarily for all $t < \tau$.
   
   (ii) (Consistency): $\varphi(t; \tau, x(\tau), \omega) = x(t)$ for all $t \in \mathbb{T}$, $x \in \mathbb{X}$, and $\omega \in \Omega$.
   
   (iii) (Composition Property): For all $t_1 < t_2 < t_3$, $x \in \mathbb{X}$, and $\omega \in \Omega$, the following condition holds: $\varphi(t_3; t_1, x(t_1), \omega) = \varphi(t_3; t_2, \varphi(t_2; t_1, x(t_1), \omega), \omega)$.
(iv) (Causality): If \( \omega, \tilde{\omega} \in \Omega \) and \( \omega_{[\tau,t]} = \tilde{\omega}_{[\tau,t]} \), then \( \varphi(t;\tau,x(\tau),\omega)) = \varphi(t;\tau,x(\tau),\tilde{\omega})) \).

(5) There exists a sensor (i.e., readout) map \( \eta : \mathbb{T} \times X \to Y \) which defines the output \( y(t) = \eta(t,x(t)) \). The map \( (\tau,t) \to Y \) given by \( \sigma \mapsto \eta(\sigma,\varphi(\sigma;\tau,x(\tau),\omega)) \) for \( \sigma \in (\tau,t) \) is an output segment, i.e., the restriction \( \gamma_{(\tau,t)} \) of some \( \gamma \in \Gamma \) to \( (\tau,t) \).

Now, the concept of finite-dimensional time-invariant dynamical systems is formally introduced below.

**Definition 4.2.** The dynamical system \( S \) in Definition 4.1 is called stationary or time-invariant if the following conditions hold:

1. \( \mathbb{T} \) is an additive group (under addition of reals).
2. \( \Omega \) is closed under the shift operator \( z^\tau : \omega \mapsto \tilde{\omega} \) defined by \( \tilde{\omega} = \omega(t+\tau) \) for all \( \tau, t \in \mathbb{T} \).
3. \( \varphi(t;\tau,x(\tau),\omega) = \varphi(t+\theta;\tau+\theta,x(\tau+\theta),z^\tau\omega) \) for all \( \theta \in \mathbb{T} \).
4. The map \( \eta(t,\bullet) : X \to Y \) is independent of \( t \).

**Definition 4.3.** The dynamical system \( S \) in Definition 4.1 is called continuous-time if \( \mathbb{T} \) is bijective to \( \mathbb{R} \), the set of reals; and \( S \) is called discrete-time if \( \mathbb{T} \) is bijective to \( \mathbb{Z} \), the set of integers.

**Definition 4.4.** The dynamical system \( S \) in Definition 4.1 is called finite-dimensional if \( X \) is a finite-dimensional vector space over a given field (e.g., \( \mathbb{R} \) or \( \mathbb{C} \)); \( S \) is called finite-state if \( X \) is a finite set; and \( S \) is called finite if \( X, \mathbb{U} \) and \( Y \) are finite sets and, in addition, \( S \) is time-invariant and discrete-time. A finite system \( X \) is often called a finite automaton.

**Definition 4.5.** The dynamical system \( S \) in Definition 4.1 is called linear if the following conditions hold:

1. Each of \( X, \mathbb{U}, \Omega, Y, \) and \( \Gamma \) is a vector space over a given field \( \mathbb{F} \) of scalars.
2. The map \( \varphi(t;\tau,\bullet,\bullet) : X \times \Omega \to X \) is \( \mathbb{F} \)-linear for all \( t \) and \( \tau \).
3. The map \( \eta(t,\bullet) : X \to Y \) is \( \mathbb{F} \)-linear for all \( t \).
Definition 4.6. The dynamical system $S$ in Definition 4.1 is called $n$-smooth for a given $n \in \mathbb{N}$ if the following conditions hold:

1. $T \subseteq \mathbb{R}$ and the topology of $T$ is diffeomorphic to the usual topology of $\mathbb{R}$.
2. Topologies are defined for $X$ and $\Omega$.
3. The transition map $\varphi$ has the property that $(\tau, x, \omega)$ defines a continuous map $T \times X \times \Omega \to C^n(T \to X)$, where $C^n(T \to X)$ denotes the family of $C^n$ functions $T \to X$.

Remark 4.1. In many applications, one may encounter dynamical systems whose system states might be hidden from experimental observations that are only available in terms of their input/output behavior in the following sense:

Given an initial event $(\tau, x(\tau))$, an input segment $\omega_{[\tau, t]}$ acting upon the dynamical system $S$ in Definition 4.1 produces an output segment $\gamma_{[\tau, t]}$, i.e., there exists a map $f_{\tau,x} : \omega_{[\tau, t]} \to \gamma_{[\tau, t]}$. Then, the output at $t \in (\tau, t]$ is given by $f_{\tau,x}(\omega_{[\tau, t]})(\tilde{t}) = \eta(t, \varphi(\tilde{t}, \tau, x(\tau), \omega))$.

Conversely, any family of functions that have the the same properties as the composition of functions $\varphi$ and $\eta$ in Definition 4.1) can be viewed to define a dynamical system in the input-output sense. A formal definition follows.

Definition 4.7. A dynamical system $S$ in the input-output sense is defined as a composite mathematical concept as follows:

1. There exist sets $T, U, \Omega, Y, \text{ and } \Gamma$, satisfying all properties required by Definition 4.1.
2. There exists a set $A$ indexing a family of functions $B = \{f_\alpha : T \times \Omega \to Y, \alpha \in A\}$, where each member of $B$ is explicitly given as: $f_\alpha(t, \omega) = y(t)$, which is the output at time $t$ from the input $\omega$ under the experiment $\alpha$. Each $f_\alpha$ is an input/output function and satisfies the following conditions:
   
   (i) (Direction of time): There exists a map $\zeta : A \to T$ such that $f_\alpha(t, \omega)$ is defined for all $t \geq \zeta(\alpha)$.
   
   (ii) (Causality): Let $\tau, t \in T$ and $\tau < t$. If $\omega, \tilde{\omega} \in \Omega$ and $\omega_{[\tau, t]} = \tilde{\omega}_{[\tau, t]}$, then $f_\alpha(t, \omega) = f_\alpha(t, \tilde{\omega})$ for all $\alpha \in A$ such that $\tau = \zeta(\alpha)$. 