Homework Set #5

Question 1: (Ergodic Hypothesis)

(a) Consider a harmonic oscillator with Hamiltonian

\[ H = \frac{1}{2}p^2 + \frac{1}{2}q^2 = \frac{1}{2}(x \cdot x) \]

Show that any phase space trajectory \( x(t) \) with energy \( E \) will, on the average, spend equal time in all regions of the constant energy surface \( \Gamma(E) \). (Hint: Use the coordinate transformation, \( p = r \cos \theta, \ q = r \sin \theta \))

(b) Consider two linearly coupled harmonic oscillators

\[ H = \frac{1}{2}(p_1^2 + p_2^2 + q_1^2 + q_2^2 + q_1q_2) \]

(i) Using the Hamiltonian dynamics, express the system in the following (state-space) form

\[ \ddot{X} = -AX \quad \text{where}, \ X = [q_1 \ q_2]^T, \ \text{and} \ A \ \text{is a} \ 2 \times 2 \ \text{matrix} \]

(ii) The present system dynamics (in the above form) has the following solution: (as the eigenvalues of \( A \), \( \lambda_1 \) and \( \lambda_2 \) are nonnegative)

\[ X = av_1 \sin(\sqrt{\lambda_1}t + \phi_1) + bv_2 \sin(\sqrt{\lambda_2}t + \phi_2) \]

where, \( v_1 \) and \( v_2 \) are the eigenvectors of \( A \) corresponding to the eigenvalues \( \lambda_1 \) and \( \lambda_2 \) respectively. \( a, b, \phi_1 \) and \( \phi_2 \) are constants and can be determined for a given initial condition. Find out the constant energy surface for the system in the space of \( a, b \).

(iii) Show that there are regions of the constant energy surface which are not visited for any particular trajectory. (Hint: For a given initial condition, find out the system trajectory on the constant energy surface computed above)

(c) Comment on the ergodic nature of both systems.

Question 2: (Poincare Recurrence and Recurrence Plots)

Consider the Logistic Map:

\[ x_{n+1} = rx_n(1 - x_n) \quad \text{with} \ r = 4 \]

(a) In this part, we will numerically investigate the Poincare Recurrence of this map. Fix an \( \epsilon \) (say, 0.01). Start with any randomly chosen initial condition \( \in (0,1) \). Note the time, the map takes return (for the first time) to the \( \epsilon \)-ball around the initial condition. Repeat the experiment with many (say, around 10,000) such random initial conditions and compute the average first return-time over the set of initial conditions. Now vary \( \epsilon \) (say, 0.01 to 0.1 with a step size 0.005) and plot the variation of average first return-time with \( 1/\epsilon \).
Let the trajectory of a one-dimensional system in its phase be \( \{x_i\} \) for time index \( i = 1, 2, \cdots, N \). Now, consider the following recurrence matrix for a given \( \epsilon > 0 \):

\[
R_{ij} = \begin{cases} 
1, & \text{if } |x_i - x_j| \leq \epsilon \\
0, & \text{if } |x_i - x_j| > \epsilon
\end{cases} \quad i, j = 1, 2, \cdots, N
\]

The Recurrence plot of such a one-dimensional system will be a graph of \( R_{ij} = 1 \) with \( i \) on horizontal axis and \( j \) on vertical axis (i.e., for a particular time instant \( i \), plot all such time instants, \( j \) when \( x_i \) and \( x_j \) are reasonably close).

(i) Draw the Recurrence plot for the system in part (a) with \( \epsilon = 0.05 \) and \( N = 200 \).

(ii) Draw the Recurrence plots for a sinusoid signal and a uniformly random signal (use the function ‘rand’ if you are using MATLAB). Keep the length of the phase space as 1 for both signals; also take \( \epsilon = 0.05 \) and \( N = 200 \). (Suggestion: Take a reasonable frequency for the sinusoid signal to have 6 to 10 cycles within \( N = 200 \))

(iii) Comment on your observations of these three Recurrence plots.

**Question 3: (Invariant Density and Markov Chains)**

![Figure 1: Plot of the Map for Question 3](image)

(a) Find the natural invariant density for the map in given in Fig. 1 assuming that \( \rho(x) \) is constant in each of the three intervals \((0, \frac{1}{3}), (\frac{1}{3}, \frac{2}{3}), (\frac{2}{3}, 1)\). (Hint: Obtain the stochastic matrix for the corresponding Markov chain)

(b) What is the fraction of time a typical orbit spends in the interval \( \frac{1}{6} \leq x \leq \frac{1}{2} \)?

Try these for fun (you don’t need to submit):

(i) Take a large number (say, 50,000) of initial states (with any distribution) and evolve them with the map equation for some time (say, 100 iterations). Does the distribution of states converge to the invariant density computed in part (a)?

(ii) Take one randomly chosen initial states and evolve that with the map equation for a long time (say, 50,000 iterations). Does the temporal distribution of the state evolution converge to the invariant density computed in part (a)?