The notion of cardinality of a set $\Omega$, denoted as $\text{card}(\Omega)$, is related to the number of elements in $\Omega$. If $\Omega$ has finitely many (say n) elements, then $\text{card}(\Omega)=n$. However, the concept is not straightforward for infinite sets.

We will consider the following three cases for two sets $X$ and $Y$ (regardless of whether they are finite or infinite):

$\text{card}(X)=\text{card}(Y)$; $\text{card}(X) \leq \text{card}(Y)$; $\text{card}(X)<\text{card}(Y)$.

**Definition A3-1:** Let $X$ and $Y$ be two nonempty sets. Then,

- $\text{card}(X)=\text{card}(Y)$ if there exists a bijective mapping between $X$ and $Y$.
- $\text{card}(X) \leq \text{card}(Y)$ if there exists an injective mapping from $X$ into $Y$.
- $\text{card}(X)<\text{card}(Y)$ if every injective mapping from $X$ into $Y$ is NOT onto, i.e., its range is a strictly proper subset of $Y$.

**Theorem A3-1:** Let $X$ and $Y$ be two sets. If $\text{card}(X) \leq \text{card}(Y)$ and $\text{card}(Y) \leq \text{card}(X)$, then $\text{card}(X)=\text{card}(Y)$.

**Proof:** It is obvious from Definition A3-1. $\blacksquare$

**Definition A3-2:** A set is called *countable* if there exists a bijective mapping between $X$ and the set, $\mathbb{N} = \{1,2,3,\cdots\}$, of natural numbers. A set which is either finite or countable is called *at most countable*. An infinite set which is not countable is called *uncountable*.

**Remark A3-1:** $\text{card}(\text{a finite set}) \leq \text{card}(\text{an at most countable set}) \leq \text{card}(\text{a countable set})=\text{card}(\text{another countable set}) < \text{card}(\text{an uncountable set})$.

**Remark A3-2:** $\text{card}(\emptyset)=0$.

**Remark A3-3:** The set, $\mathbb{J}$, of all integers and the set, $\mathbb{N}$, of all positive integers belong to the same class of cardinality even though $\mathbb{N}$ is a proper subset of $\mathbb{J}$. This fact may appear to be counter-intuitive from the perspective of finite sets.

**Example A3-1:** To show that the set, $\mathbb{J}$, of all integers is countable, i.e., $\text{card}(\mathbb{Z})=\text{card}(\mathbb{N})$, we find a bijective mapping $f : \mathbb{N} \to \mathbb{J}$ as follows:

$$f(n) = \begin{cases} \frac{n}{2} & \text{for } n \text{ even} \\ -\frac{n-1}{2} & \text{for } n \text{ odd} \end{cases}$$

Similarly, to show that the set, $\mathbb{Z} = \{0,1,2,\cdots\}$ of non-negative integers is countable, i.e., $\text{card}(\mathbb{Z})=\text{card}(\mathbb{N})$, we find a bijective mapping $f : \mathbb{N} \to \mathbb{Z}$ as follows:

$$f(n) = n-1.$$

**Theorem A3-2:** Let $A$ be a countable set and $B_n$ be the set of all $n$-tuples $(a_1,a_2,\cdots,a_n)$ where $a_k \in A$. Then, $B_n$ is countable.

**Proof:** For $n=1$, $B_1 = A$ is countable. Suppose $B_{n-1}$ is countable for some $n>1$. Then, the elements of $B_n$ are of the form $(b,a)$ where $b \in B_{n-1}$ and $a \in A$. For every fixed $b$, the set of pairs $(b,a)$ bears an equivalence relation
with \(A\), and hence is countable. Therefore, \(B_n\) is the union of a countable set of countable sets. By Theorem A3-2, \(B_n\) is countable. Now, the proof is completed by induction.

**Corollary to Theorem A3-2:** The set, \(Q = \{\frac{m}{n} : m \in J; n \in J - \{0\}\}\) of rational numbers is countable.

**Theorem A3-3:** Let \(\{s_i\}\) be a sequence of countable sets, i.e., \(s_i\) is a countable set for every \(i = 1, 2, 3, \ldots\). Then, a countable union of \(s_i\)'s is also a countable set.

**Proof:** See Rudin (p. 29).

**Theorem A3-4:** The set \((0,1) = \{x : 0 < x < 1\} \subset \mathbb{R} = (-\infty, \infty)\) is uncountable.

**Proof:** Let us assume that the set \((0,1)\) is countable, i.e., \((0,1) \sim N\). That is, we assume the existence of a bijective mapping between the sets \((0,1)\) and \(N\), in which every \(x_i \in (0,1)\) corresponds to a unique \(i \in N\). Let us consider a real number \(x_i \in (0,1)\) that is written as \(0.\delta_1\delta_2\delta_3\delta_4\ldots\) where \(0 \leq \delta_i \leq 9\) is an integer. We define \(\tilde{\delta}_i = 9 - \delta_i\) so that \(0 \leq \tilde{\delta}_i \leq 9\). Consider the real number \(y = 0.\tilde{\delta}_1\tilde{\delta}_2\tilde{\delta}_3\tilde{\delta}_4\ldots\) that certainly belongs to the set \((0,1)\). Since \(y\) must differ from any of the \(x_i\)'s defined above, the bijective mapping from \(N\) onto \((0,1)\) is impossible. Therefore, the above assumption that the set \((0,1)\) is countable is false.

**Remark A3-4:** The sets \(N\) and \(\mathbb{R}\) belong to the different classes of cardinality.

**Remark A3-5:** The sets \((0,1)\) and \(\mathbb{R}\) have the same cardinality. This will be clear after we study the topological spaces. We will show that \((0,1)\) and \(\mathbb{R}\) are homeomorphic under the usual topology. Loosely speaking, this means that the sets \((0,1)\) and \(\mathbb{R}\) are indistinguishable from the topological perspectives. Candidate mappings are: \(f: (0,1) \rightarrow (-\infty, \infty)\) with \(f(x) = \tan^{-1}\left(\frac{x - \frac{1}{2}}{1}\right)\) and \(f(x) = \frac{2x - 1}{x(x - 1)}\), which are both bijective and bicontinuous. Therefore, \(f\) is a homeomorphism. Let us look at the picture from a geometric point of view in the diagram below.

Represent \(\mathbb{R}\) by an infinite straight line axis on which each point represents a unique real number. Now draw the following figure after bending the line segment \((0,1)\) into a semicircle. If the lines are drawn from the center of the semicircle intersecting both the semicircle and the infinite straight line axis, the points of intersection can be paired as a bijection from \((0,1)\) onto \(\mathbb{R}\).

**Zorn’s Lemma**

**Definition A3-3:** A relation \(\leq\) on a nonempty set \(S\) is said to be a partial ordering if

- \(x \leq x\) \(\forall x \in S\)
- \((x \leq y\text{ and } y \leq x) \Rightarrow x = y\) \(\forall x, y \in S\)
- \((x \leq y\text{ and } y \leq z) \Rightarrow x \leq z\) \(\forall x, y, z \in S\)

A set \(S\) is said to be partially ordered if \(S\) has a defined partial ordering.

**Definition A3-4:** A partial ordering \(\leq\) on a nonempty set \(S\) is said to be a total ordering if, in addition,

- either \(x \leq y\) or \(y \leq x\) \(\text{ for any two points } x, y \in S\)
A set $S$ is said to be totally ordered if $S$ has a defined total ordering.

**Definition A3-5:** Let $\leq$ be a partial ordering on a nonempty set $S$ and let $A \subseteq S$ be nonempty. Then, the set $A$ is said to be a chain if $A$ is totally ordered, i.e., either $x \leq y$ or $y \leq x$ for any two points $x, y \in A$.

**Definition A3-6:** Let $\leq$ be a partial ordering on a nonempty set $S$ and let $A \subseteq S$ be nonempty. Then, $\tilde{a} \in S$ is said to be an upper bound of $A$ if $y \leq \tilde{a} \quad \forall y \in A$. Furthermore, if $\tilde{a} \in A$, then $\tilde{a}$ is a maximal element of $A$.

**Remark A3-6:** A chain is a restriction of a partial ordering that yields a total ordering.

**Example A3-2:** The collection $\mathcal{P}$ of all open subsets of $\mathbb{R} \times \mathbb{R}$ is partially ordered but not totally ordered.

**Example A3-3:** The collection $\mathcal{T}$ of all open disks in $\mathbb{R} \times \mathbb{R}$ is totally ordered. Furthermore, $\mathcal{T} \subseteq \mathcal{P}$ is a maximally totally ordered set, i.e., if any member of $\mathcal{P}$ which is not in $\mathcal{T}$ is adjoined with $\mathcal{T}$, then the resulting collection of sets is no longer totally ordered by $\subseteq$.

**Theorem A3-5 (Zorn’s Lemma):** If every chain in a partially ordered set $S$ has an upper bound, then $S$ has a maximal element.

**Remark A3-7:** Zorn’s Lemma can be interpreted as follows: In a nonempty set $c$ with partial ordering $\leq$, let every chain $U \subseteq S$ have an upper bound, i.e., $\exists x \in S$ s.t. $x \geq \alpha \quad \forall \alpha \in U$. Then, $S$ has a maximal element. (Equivalent to Zorn’s Lemma.)

**Theorem A3-7 (Axiom of Choice):** Let $I$ be an index set for a partially ordered set $S$ and $\exists$ a nonempty $S_\alpha \subseteq S \quad \forall \alpha \in I$. Let $\Sigma$ be collection of all such functions $S_\alpha$. Then, one can define a function $\vartheta : I \rightarrow \Sigma$.

**Theorem A3-6 (Hausdorff Maximality Theorem):** Every partially ordered set contains a maximally totally ordered set. (Equivalent to Zorn’s Lemma.)

**HW#A3-1:** Show that, given real numbers $a$ and $b$, $\{x^2 + ax + b \geq 0 \text{ for all } x \in \mathbb{R}\} \Leftrightarrow \{a^2 - b \leq 0\}$.

**HW#A3-2:** Show that $\sqrt{3}$ is irrational.

**HW#A3-3:** Let $x \in \mathbb{R}$; $f(x) = x$; and $g(x) = e^x$. Show that $f(0) < g(0)$ and $f'(x) < g'(x)$ $\forall x > 0$. Now show that $x \geq 0 \Rightarrow x < e^x$.

**HW#A3-4:** $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ for $n \in \mathbb{N}$. Then, evaluate $\sum_{k=1}^{n} k^2$ and $\sum_{k=1}^{n} k^3$. Use different methods for proof.

**HW#A3-5:** Show that, given $h \in (0, \infty)$, show that $(1 + h)^n > (1 + nh) \quad \forall n \in \mathbb{N} - \{0\}$.

**HW#A3-6:** Show that a total ordering on a set is an equivalence relation. Is the converse true?