Today, we will:

- Review the pdf module: **Dynamic System Response (1\textsuperscript{st}-order systems)**
- Do some example problems – dynamic system response for first-order systems

**Example: First-order dynamic system response**

Given: A first-order low-pass filter with $R = 100\ \text{k}\Omega$ and $C = 0.010\ \mu\text{F}$

(a) **To do:** Calculate the time constant $\tau$ and the static sensitivity $K$ of this system.

(b) **To do:** Discuss how time constant $\tau$ is related to the cutoff frequency of the filter.

(c) **To do:** For a sudden change in input voltage, how long will it take for the nondimensional output to reach 99% of its final value?

(d) **To do:** If $y_i = 1\ \text{V}$ and $y_f = 3\ \text{V}$, calculate $y$ when $t = 10.0\ \text{ms}$.

**Solution:**

(a) From the learning module, we write the \texttt{1\textsuperscript{st}-order ODE for a low-pass filter}:

- By definition, $K = \frac{b}{a_0} = \frac{1/R}{1/R} = 1$
  - $K = 1 = \text{static sensitivity}$
  
- $\zeta = \frac{a_i}{a_0} = \frac{C}{1/R} = RC$
  - $\zeta = RC = \text{1\textsuperscript{st}-order time constant}$

- Numbers: $\zeta = (100,000\ \text{pF})(0.01\ \times 10^{-6}\ \text{F})\left(\frac{C}{V\cdot\text{F}}\right)\left(\frac{A}{\text{V}}\right)\left(\frac{\text{V}}{\text{A}\cdot\text{sec}}\right) = 0.0010\ \text{s}$
  - $\zeta = 1.0\ \text{ms}$
(b) Compare \( \omega \) to cutoff frequency of the filter:

Recall, \( f_{\text{cutoff}} = \frac{1}{2\pi RC} \rightarrow \omega = \frac{1}{2\pi \tau} \)
\[ f_{\text{cutoff}} = \frac{1}{2\pi \tau} \]
\[ \omega_{\text{cutoff}} = \frac{1}{RC} = \frac{1}{\tau} \]

Numbers:
\[ f_{\text{cutoff}} = \frac{1}{2\pi (0.0010 \text{ s})} = 159 \text{ Hz} \]

(c) How long to reach 99\% of final value?

![Diagram showing input sudden change and output behavior](image)

- Use the nondimensional form of the solution:
  
  \[ \frac{Y - Y_c}{Y_f - Y_c} = 1 - e^{-t/\tau} \]
  
  Set to 0.99

- Equation from the learning module, nondimensional (holds for any \( y_i \) and \( y_f \))

\[ \frac{Y - Y_c}{Y_f - Y_c} = 1 - e^{-t/\tau} \]

Solve for \( t \):
\[ 0.99 = 1 - e^{-t/\tau} \]
\[ 0.01 = e^{-t/\tau} \]
\[ \ln(0.01) = -t/\tau \]
\[ t = -\tau \ln(0.01) = -(0.0010 \text{ s}) \ln(0.01) \]
\[ t = 0.00461 \text{ s} \]

**Note:** Since \( \tau = 0.0010 \text{ s} \), the \( t \) is 4.61 time constants, close to 5 time constants.
Most people round to 5 ± 99% → A first-order system needs about 5 time constants to get within 99% of the final change of response

\[\text{For a sudden change}\]

(2) If \(y_i = 1V\) & \(y_f = 3V\), calculate \(y\) at \(t = 10\) ms

Again, we use the general nondimensional equation since it applies to any \(y_i\) & \(y_f\)

\[
\frac{y - y_i}{y_f - y_i} = 1 - e^{-\frac{t}{\tau}} \rightarrow y = y_i + (y_f - y_i)(1 - e^{-\frac{t}{\tau}})
\]

\[\text{Numbers:} \quad y = 1V + (3V - 1V)(1 - e^{-\frac{10\text{ ms}}{1\text{ ms}}}) = 2.999998 V \approx 3.00 V\]

[After 10 time constants we have “reached” the final response output to the 5th digit]

The general nondimensional equation applies even when the step is down

\[\text{E.g. Let } y_i = 3V, y_f = 1V\]

\[\frac{y - y_i}{y_f - y_i} = 1 - e^{-\frac{t}{\tau}} \rightarrow y = y_i + (y_f - y_i)(1 - e^{-\frac{t}{\tau}})\]

\[y = 3V + (1V - 3V)(1 - e^{-\frac{10\text{ ms}}{1\text{ ms}}})\]

\[y = 1.7358 V\]

[After \(t = \text{one time constant (}t = \tau)\), \(y\) drops from 3 V to 1.7358 V]
Example: First-order dynamic system response

Given: A thermometer behaves as a first-order dynamic system with time constant \( \tau = 1.00 \text{ s} \). At \( t = 0 \), the thermometer is plunged from a tank of ice water at \( T = 0^\circ \text{C} \) into a tank of boiling water at \( T = 100^\circ \text{C} \).

To do: Calculate how long (in seconds) it takes for the thermometer to read \( 50^\circ \text{C} \).

Solution:

\[
\frac{y - y_i}{y_f - y_i} = 1 - e^{-t/\tau}
\]

At \( T = 50^\circ \text{C} \),

\[
\frac{y - y_i}{y_f - y_i} = \frac{50 - 0}{100 - 0} = 0.5
\]

Set this to 0.5 and solve for \( t \):

\[
0.5 = 1 - e^{-t/\tau}
\]

\[
-0.5 = -e^{-t/\tau}
\]

\[
\ln(0.5) = \ln(e^{-t/\tau}) = -t/\tau
\]

\[
t = -\tau \ln(0.5) = -(1.00 \text{ s}) \ln(0.5) = 0.69315 \text{ s}
\]

\( t \approx 0.693 \text{ s} \)

By the way, this is called the half-life \( t_{1/2} \).

Time required to go half way from \( y_i \) to \( y_f \).
Example – A practical example of 1st-order dynamic system analysis

Given: A fire has occurred in a hotel room with volume $V = 85 \, \text{m}^3$. Immediately after the fire is extinguished, the mass concentration of hydrogen cyanide (HCN) is $c_i = 10,000 \, \text{mg/m}^3$. The firemen blow “fresh” air into the room at $\dot{V} = 28.3 \, \text{m}^3/\text{min}$. Since there is some smoke outside the building, the “fresh” air actually has an ambient mass concentration of HCN equal $c_a = 1.0 \, \text{mg/m}^3$. The air is considered safe when the mass concentration of HCN in the room drops below $c = 5 \, \text{mg/m}^3$.

To do: Calculate how long the firemen need to wait before entering the room.

Solution: Room ventilation is modeled by using a first-order ODE for mass concentration $c$ in the room:

$$\frac{dc}{dt} = -\frac{V}{V} c + \frac{V}{V} c_a$$

Compare to standard form:

- $a_i = 1$
- $a_0 = \frac{\dot{V}}{V}$
- $b = \frac{V}{V}$
- $y = c$
- $x = c_a$

Non-dimensional equation is:

$$\frac{y-y_i}{y_f-y_i} = 1-e^{-\frac{t}{\tau}}$$

Solve for $t$:

$$t = -\frac{\tau}{\tau} \ln \left(1 - \frac{y-y_i}{y_f-y_i}\right) = -\frac{\tau}{\tau} \ln \left(1 - \frac{c-c_i}{c_f-c_i}\right)$$

Where $C = \text{the “safe” concentration that we desire}$ (5)

- $C_f = C_a = \text{ambient concentration}$ [as $t \to \infty$, $C \to C_a = \text{ambient concentration}$] (1)
- $C_i = \text{initial concentration in the room} (10,000)$

Number:

$$t = \frac{-85 \, \text{m}^3}{28.3 \, \text{m}^3/\text{min}} \ln \left(1 - \frac{5 - 10,000}{1 - 10,000}\right) = 23.5 \, \text{minutes}$$

Answer to 2 digits -> $t = 24 \, \text{minutes}$

[The firemen must wait $\frac{1}{2} \, \text{hr. to enter safely}$.]
Note: \[ C = \frac{A}{V} = \frac{85 \text{ m}^3}{28.3 \text{ m}^3/\text{min}} = 3.0035 \text{ min} \]

Need: \[ \frac{23.5}{3.0035} = 7.8 \text{ TJs} \]

The room needs 2.8 time constants for C to decrease to safe levels.