How to iterate to find \( v_r \) (terminal radial speed for inertial separation)

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**Example** (this example is for inertial separation; a similar procedure is used for gravimetric settling)

**Given:**
- Spherical particles of 2.5 µm diameter with particle density = 1600 kg/m³.
- Inertial separation with \( U_\theta = 10.0 \) m/s and average radius of curvature \( r_m = 0.25 \) m.
- Air at STP: \( \rho = 1.184 \) kg/m³ and \( \mu = 1.849 \times 10^{-5} \) kg/(m s).

**To do:** Calculate the terminal radial speed for these particles:

**Solution:** The equations we need are:

\[
Kn = \frac{\lambda}{D_p}, \quad \lambda = \frac{\mu}{0.499 \sqrt{8 \rho P}}, \quad C = 1 + Kn \left[ 2.514 + 0.80 \exp \left(-\frac{0.55}{Kn}\right) \right], \quad C_D = C_D(Re), \quad \text{where} \quad Re = \frac{\rho v_r D_p}{\mu}.
\]

There are many equations for the drag coefficient of a sphere: \( C_D = \frac{24}{Re} \) for \( Re < 0.1 \) (Stoke’s flow), \( C_D = \frac{24}{Re}(1 + 0.0916 Re) \) for \( any \) \( Re < 5 \), \( C_D = \frac{24}{Re}(1 + 0.158 Re^{1/2}) \) for \( 5 < Re < 1000 \), \( C_D = 0.4 + \frac{24}{Re} + \frac{6}{1 + \sqrt{Re}} \) for \( 1000 < Re < 10^5 \), and \( C_D \approx 0.2 \) for \( Re > 2 \times 10^6 \). Note that \( C_D \) is a strong function of surface roughness for \( Re \) between about \( 10^5 \) and \( 10^6 \).

Also, the equations for terminal settling speed and terminal radial speed are:

- **Gravimetric settling:**
  \[
  V_t = \frac{4 \rho_p - \rho}{3 \rho} g D_p C_D \quad \text{Re} = \frac{\rho V_t D_p}{\mu}
  \]
- **Inertial separation:**
  \[
  v_r = \frac{4 \rho_p - \rho}{3 \rho} U_\theta^2 D_p r_m C D_D \quad \text{Re} = \frac{\rho v_r D_p}{\mu}
  \]

**Note:** For inertial separation, we substitute the radial acceleration \( U_\theta^2/r_m \) in place of gravitational acceleration \( g \), and we substitute the radial speed \( v_r \) in place of settling speed \( V_t \). Other than that, the equations and procedure are identical for either gravimetric settling or inertial separation.

In our example, \( \lambda = \frac{\mu}{0.499 \sqrt{8 \rho P}} = \frac{1.849 \times 10^{-5} \text{ kg/(m s)}}{0.499 \sqrt{8(1.184 \text{ kg/m}^3)(101325 \text{ N/m}^2)}} \left( \frac{10^6 \text{ µm}}{\text{m}} \right) = 0.06704 \text{ µm} \),

\[
Kn = \frac{\lambda}{D_p} = \frac{0.06704 \text{ µm}}{2.5 \text{ µm}} = 0.026816, \quad \text{and} \quad \text{Cunningham} = C = 1 + 0.026816 \left[ 2.514 + 0.80 \exp \left(-\frac{0.55}{0.026816}\right) \right] = 1.0674.
\]

Now we need to set up our iteration. First we guess \( v_r \). Stokes approximation is typically a good first guess, but you can pick any guess you want – it just might take longer to converge. Stoke’s approximation is

\[
V_t = \frac{\rho_p - \rho}{18} D_p g \frac{C}{\mu}
\]

for gravimetric settling, and we substitute \( U_\theta^2/r_m \) in place of \( g \) for inertial separation,

\[
v_r = \frac{\rho_p - \rho}{18} D_p^2 \left( \frac{U_\theta^2}{r_m} \right) \frac{C}{\mu}.
\]

So, for our problem at hand, a good first guess is

\[
v_r = \frac{\rho_p - \rho}{18} D_p^2 \left( \frac{U_\theta^2}{r_m} \right) \frac{C}{\mu} = \frac{(1600 - 1.184) \text{ kg/m}^3}{18} \left( 2.5 \times 10^{-6} \text{ m} \right)^2 \left( \frac{10.0 \text{ m/s}^2}{0.25 \text{ m}} \right) \left( 1.849 \times 10^{-5} \text{ kg/(m s)} \right) = 0.012819 \text{ m/s}.
\]

This Stokes value is usually used as a first guess. However, the procedure should still converge even if we pick a different first guess.
Here we show how we can converge on the correct answer when we pick 0.02 m/s as our first guess. We set up a table to iterate, using the following equations in sequence, and then using the new $v_r$ as our next guess:

$$Re = \frac{\rho v_r D_p}{\mu}, \quad C_D = \frac{24}{Re} \left(1 + 0.0916Re\right), \quad v_r = \sqrt[3]{\frac{4 \rho_p - \rho U_0^2}{3 \rho} \frac{C}{r_m} D_p C_D}$$

This is most easily done in Excel, but I show it here “by hand” in a table:

<table>
<thead>
<tr>
<th>$v_r$ (m/s)</th>
<th>Re</th>
<th>$C_D$</th>
<th>New $v_r$ (m/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.02</td>
<td>0.0032017</td>
<td>7498.14</td>
<td>0.01601</td>
</tr>
<tr>
<td>0.01601</td>
<td>0.002563</td>
<td>9366.47</td>
<td>0.01432</td>
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<td>0.02293</td>
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</tr>
<tr>
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<td>0.002169</td>
<td>11066.8</td>
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<tr>
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<td>0.0021096</td>
<td>11378.7</td>
<td>0.012996</td>
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<td>0.012906</td>
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<td>0.0020536</td>
<td>11689.1</td>
<td>0.012822</td>
</tr>
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<td>11689.1</td>
<td>0.012822</td>
</tr>
</tbody>
</table>

We see that we have converged to 5 significant digits in $v_r$. We note that 3 significant digits is probably about the best we can hope for in this kind of exercise. Nevertheless, we write the final values to 4 significant digits below:

$$Re = 0.002054, \quad C_D = 1169, \quad v_r = 0.01282 \text{ m/s}$$