Today, we will:

- Demonstrate how to set up particle trajectory calculations in a 2-D air flow using Runge-Kutta
- Discuss **equivalent diameter** for non-spherical particles
- Discuss **gravimetric settling** in rooms and ducts [Section 8.7]
- Do **Candy Questions** for **Candy Friday**

Summary from last time: the equation of particle motion in a (known) air flow is

\[
\frac{\pi D_p^3}{6} \frac{d\vec{v}}{dt} = \frac{\pi D_p^3}{6} (\rho_p - \rho) \vec{g} - \frac{\rho C_D \pi D_p^{2/3}}{4} v_r |v_r| \]

Multiply all terms by \( \frac{6}{\pi D_p^3 \rho_p} \). The above equation (Newton’s law) reduces to

\[
\frac{d\vec{v}}{dt} = \frac{\rho_p - \rho \vec{g}}{\rho_p} - \frac{3 \rho C_D}{4 \rho_p C D_p^2} v_r |v_r| \]

where

- \( \vec{v} \) is the particle velocity
- \( \vec{U} \) is the air velocity
- \( v_r = \vec{v} - \vec{U} \) is the relative particle velocity
- \( \rho \) is the air density
- \( \rho_p \) is the particle density
- \( D_p \) is the particle diameter
- \( C \) is the Cunningham correction factor
- \( C_D \) is the drag coefficient, which is a function of Reynolds number, \( \text{Re} = \frac{\rho |v_r| D_p}{\mu} \)

For simplicity, consider only 2-D flow in the \( x-y \) plane

\( (x, y) = \text{coord.} \quad (\hat{i}, \hat{j}) = \text{unit vector} \)

Particle velocity = \( \vec{v} = (v_x, v_y) \)

Air velocity = \( \vec{U} = (U_x, U_y) \)

\( \vec{v}_r = \vec{v} - \vec{U} \)

\( \vec{v}_r = (v_x - U_x) \hat{i} + (v_y - U_y) \hat{j} \)

\( v_r = |\vec{v}_r| = \sqrt{(v_x - U_x)^2 + (v_y - U_y)^2} \)
In (6) \[ \dot{V}_r \mid v_r \mid = (V_x - U_x) v_r \hat{i} + (V_y - U_y) v_r \hat{j} \]

(6) Split into \( x \) and \( y \) components:

\[
\begin{align*}
\dot{x} & = \frac{dV_x}{dt} = 0 - \frac{3}{4} \frac{f}{p_p} \frac{C_0}{C \, D_p} \left( V_x - U_x \right) v_r \\
\end{align*}
\]

(7)

\[
\begin{align*}
\dot{y} & = \frac{dV_y}{dt} = -\frac{f}{p_p} p_{p} g - \frac{3}{4} \frac{f}{p_p} \frac{C_0}{C \, D_p} \left( V_y - U_y \right) v_r \\
\end{align*}
\]

(8)

Where \( C_0 = C_0 \left( Re \right) \) : \( Re = \frac{\rho D_p v_r}{\mu} \)

Rearrange (7) and (8):

\[
\begin{align*}
\frac{dV_x}{dt} & = \frac{3}{4} \frac{f}{p_p} \frac{C_0}{C \, D_p} v_r U_x - \frac{3}{4} \frac{f}{p_p} \frac{C_0}{C \, D_p} V_r V_x \\
\frac{dV_y}{dt} & = \frac{3}{4} \frac{f}{p_p} \frac{C_0}{C \, D_p} v_r U_y - \frac{f}{p_p} p_{p} g - \frac{3}{4} \frac{f}{p_p} \frac{C_0}{C \, D_p} V_r V_y \\
\end{align*}
\]

These are standard form for 1st order ODE:

\[
\begin{align*}
\frac{dV_x}{dt} & = B_x - A_x V_x \\
\frac{dV_y}{dt} & = B_y - A_y V_y \\
\end{align*}
\]

(9)

(10)
Use Runge-Kutta solution → goal Track (follow)

$x, y$ location of the particle with time

$\vec{U} = \text{specified} \quad \vec{U} = \vec{u}(x, y, t)$

(@ air flow)

We will solve $\vec{U}$ simultaneously using $R-K$ (march in time)

Variable $1 = Y_1 = X = x$-location of particle

$D_1 = \frac{dx}{dt} = V_x = Y_2$

Variable $2 = Y_2 = V_x = x$-velocity

$D_2 = \frac{dv_x}{dt} = B_x - A_x V_x$ (Eq 9)

Variable $3 = Y_3 = Y = y$-location of particle

$D_3 = \frac{dy}{dt} = V_y = Y_4$

Variable $4 = Y_4 = V_y = y$-velocity

$D_4 = \frac{dv_y}{dt} = B_y - A_y V_y$ (Eq 10)

To Run, start at $t=0$

$x, y, V_x, V_y$ may be known at $t=0$

/ Initial condition

$Y_1, Y_2, Y_3, Y_4$

Now we march in time using R-K. I did it in Matlab.

See HW 10 — you will do this R-K problem.
Example: Direction of drag force

**Given:** A small particle is moving through the air at velocity $\vec{v}$ as sketched to the right. At the location of the particle, and at a given instant in time, the air velocity is $\vec{U}$ as also sketched.

**To do:** Which is the correct direction of the drag force on the particle?

---

**Solution:**

Do a vector summation. (c) is the correct answer.
Non-spherical particles — Equivalent diameter

(all our eqs are for sphere of dia. \(D_p\))

Actual particle \(P_p\) \(\rightarrow\) Sphere \(D_p\), equivalent

Three types of equiv. dia.

1. \(D_{ae} =\) Aerodynamic equivalent diameter
   
   \(=\) Dia. of a spherical particle of unit density
   
   that has the same terminal settling speed \(V_t\)
   
   as the actual particle

Procedure:

- Measure \(V_t\) of \(P_p\) particle
- Real \(P_p\) \(\neq\) unit dens.
- Use our eq for \(V_t\) for a sphere: set \(\rho = 1000 \text{ kg/m}^3\)

\[
V_t = \sqrt{\frac{4}{3}} \frac{Lef\rho g D_p C}{C_D}
\]

Thus \(D_p = D_{ae}\)

Need to adjust because \(C\)

Cunningham default on \(D_{ae}\)
2. $D_{sc} = \text{Spherical particle diameter} \quad \text{(Dia. of a spherical particle of the same density)}$

= Dia. of a sphere of the same terminal settling speed as the actual particle.

Procedure same as above but $p_p = p_p$ actual particle

(need to note)

3. $D_{ve} = \text{Volume equivalent Diameter} \quad \text{(Dia. of a sphere of the same volume)}$

Procedure: count $N$ particles, i. weish them = $M_{\text{total}}$

$\rightarrow m_p = \text{avg. mass/ per particle} = \frac{M_{\text{total}}}{N}$

$\rightarrow V_p = \text{avg. particle volume} = \frac{m_p}{p_p}$

Sphere: $V_p = \frac{\pi}{6}D_{ve}^3 \rightarrow$ solve for $D_{ve}$ $D_{ve} = \left(\frac{6V_p}{\pi}\right)^{\frac{1}{3}}$ (no iteration required)