Scaling and RG for spin systems

There is a deep connection between the statistical mechanics of spin chains and dynamical processes of certain kinds. Here we discuss the behavior at or near phase transitions.

We will find:

- universality
- self similarity (at the critical point)
- non-trivial power-laws (analogous to dimensions)
Exponents not quite correct
Exponents deficiencies
Exponents too universal; all not dependent on $\eta$
Exponents sufficient for many purposes

Given power-law behaviors

Success
Gives phase transitions

* from liquid-gas transition

<table>
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<tr>
<th>$\eta$</th>
<th>$T$</th>
<th>$\beta$</th>
<th>$\nu$</th>
<th>$M_F$</th>
<th>$\text{Ising 2D}$</th>
<th>$\text{Ising 3D}$</th>
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How good is MFT?
Scaling

Two observable quantities depend on each other in a power law fashion. Often the exponent can be determined by dimensional analysis.

Dynamical situation:
- Acceleration \( \frac{d^2r}{dt^2} = \text{const} \Rightarrow r(t) \sim t^{1/2} \)
- Diffusion \( \nabla^2 \Theta \sim \frac{\partial \Theta}{\partial t} \Rightarrow r(t) \sim t^{1/2} \)
- Kepler's law \( \frac{d^2r}{dt^2} \sim \frac{1}{r^2} \Rightarrow r(t) \sim t^2 \)

Exponents reveal the nature of dynamics.

Static mass \( \sim \{ \begin{array}{c} \text{line} \\ \text{surface} \\ \text{volume} \end{array} \) \)

Random walk: Each step a, after N steps \( r \sim a N^{1/2} \)

(weight \( \sim N \sim r^2 \))

Exponents are rational fractions?
Anomalous scaling
Exponents cannot always be
determined by dimensional analysis
and don't have to be simple.

Sierpinski gasket

How does mass scale with $R$?

Suppose $M \sim R^x$

$R \rightarrow 2R$, $M \rightarrow 3M$

$M(2R) = \frac{(2R)^x}{M(R)} = 2^x = 3$

$\Rightarrow x = \frac{\ln 3}{\ln 2} = 1.588 \ldots$

- The object is self-similar.
  Self similarity and scaling go hand in hand.

Self-avoiding random walk

$R \sim N 0.588 \pm 0.001$

From exp's in dilute polymer solns

$R \sim N 0.586 \pm 0.004$
Scaling relations between exponents

\[ m \propto t^\beta \quad \chi \propto t^{-\gamma} \quad h \propto m^\delta \]

Not all three exponents are independent, however.

\[ \beta \delta = \beta + \gamma \]

 guessed from numerical

<table>
<thead>
<tr>
<th>Theory</th>
<th>( \beta )</th>
<th>( \delta )</th>
<th>( \gamma )</th>
<th>( \beta \delta )</th>
<th>( \beta + \delta )</th>
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(There are other exponents and more scaling relations, which will not be discussed here.)
Scaling hypothesis (Widom)

Eq'n of state

Weiss theory \[ m = F(h,t) \]

(here \( h = \frac{H}{T} \))

\[
\begin{align*}
\text{Exp.} & \quad m(t, h=0) = \begin{cases} 
0, & t > 0 \\
\pm A |t|^\beta, & t < 0
\end{cases} \\
m(t=0, h) = \pm B |h|^{1/\xi}
\end{align*}
\]

Widom postulated

\[
\frac{M}{|t|^\beta} = F^+ \left( \frac{h}{|t|^\Delta} \right), \quad + \text{ for } t > 0
\]

\[
\frac{M}{|t|^\beta} = F^- \left( \frac{h}{|t|^\Delta} \right), \quad - \text{ for } t < 0
\]

Valid for \(|h|, |t| < 1\), but \( \frac{h}{t} \) can be anything.

Data collapse!

Only two curves!
• The values of $T_c, \Delta, \beta$ are not known a priori - must be fitted.

• They are obtained by achieving data collapse.

• Thus, even though $m$ and $t$ seem natural to us, nature chooses $\frac{m}{|t|^\beta}$ and $\frac{\hbar}{|t|^\Delta}$ as the natural variables.

• Scaling is nothing more than deciding what to plot against what!
Derivation of scaling relations

(i) \( m(t, h) = -m(t, -h) \)
\[ \Rightarrow |t|^\beta F^\pm\left(\frac{h}{|t|\Delta}\right) = -|t|^\beta F^\pm\left(-\frac{h}{|t|\Delta}\right) \]
\[ \Rightarrow F^\pm(x) = -F^\pm(-x) \]
Scaling \( F \) is an odd fun.

(ii) \( m(t, h) = |t|^\beta F^\pm\left(\frac{h}{|t|\Delta}\right) \)
For \( h=0 \) \( m(t) = |t|^\beta F^\pm(0) \)
\[ \Rightarrow F^+(0) = 0, F^-(0) = \text{non-zero constant} \]
\( \beta \) has the standard meaning.

(iii) \( \chi \bigg|_{h=0} = \frac{2m}{\partial h} \bigg|_{h=0} = |t|^{\beta-\Delta} F^\pm(0) \)
Assume \( F^\pm(0) \neq 0 \)
\[ \Rightarrow \beta - \Delta = -\gamma \Rightarrow \Delta = \beta + \gamma \]

(iv) What happens for \( t=0 \)?
Assume \( F^\pm(x) \sim x^\lambda \) as \( x \to \infty \).
\( m(0, h) \sim |t|^\beta \left(\frac{h}{|t|\Delta}\right)^\lambda = |t|^{\beta-\Delta} h^\lambda \)
\[ m(0,h) = |t|^{\beta - \Delta \lambda} h \]

For \( \beta - \Delta \lambda \), \( t \to 1 \), \( m(0,h) \to 0 \). Not acceptable

\[ \Rightarrow \beta = \Delta \lambda \]

\[ \Rightarrow m(0,h) \sim h = h^{\frac{\beta}{\Delta}} = h \]

\[ \Rightarrow \frac{\beta}{\Delta} = \frac{1}{S} \Rightarrow \Delta = \beta S \]

Combine with \( \Delta = \beta + \gamma \)

\[ \Rightarrow \beta S = \beta + \gamma \text{ non-trivial relation} \]

Does it work?

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Scaling hypothesis for free energy:

More general.

Generally

\[ f_s(t, h, g_1, g_2, \ldots) = (t \vert \Delta \vert^h, \frac{g_1}{t \vert \Delta \vert}, \frac{g_2}{t \vert \Delta \vert}, \ldots) \]

\[ \downarrow \text{singular part} \]

For now

\[ f_s(t, h) = (t \vert \Delta \vert^h) \]

\( \Delta \) is the specific heat exponent.

\[ C_v \sim \frac{\partial^2 f_s}{\partial t^2} \bigg|_{h=0} \sim |t|^{-\alpha} \]

\[ M \sim -\frac{\partial f_s}{\partial h} \bigg|_{h=0} \sim (t \vert \Delta \vert^h) \]

\[ \Rightarrow \beta = 2 - \alpha - \Delta \]

Use \( \Delta = \beta s = \beta + \gamma \) (derived earlier)

\[ \Rightarrow 2\beta + \alpha + \gamma = 2 \]
Isothermal susceptibility

\[ \chi_T \sim \frac{\partial m}{\partial h} \sim \frac{2}{\hbar} \left[ \frac{1}{|t|^{2-\alpha-\Delta}} g' \left( \frac{\hbar}{|t|\Delta} \right) \right] \]

\[ \sim \left( \frac{1}{|t|^{2-\alpha-2\Delta}} g'' \left( \frac{\hbar}{|t|\Delta} \right) \right) \]

\[ h \to 0 \quad 2-\alpha-2\Delta \]

\[ \sim |t| \]

\[ \Rightarrow -\gamma = 2-\alpha-2\Delta \quad (1) \]

Combine with \( \beta = 2-\alpha-\Delta \quad (2) \)

\[ \Rightarrow \alpha + 2\beta + \gamma = 2 \]

( Can also get \( \Delta = \beta + \gamma \). )
RG: First glance

A quick derivation of scaling
Kadanoff's ideas

Basic idea: Eliminate microscopic (short distance/short wavelength/large momentum) degrees of freedom.

- Ask close at $T_c$, the system is self-similar. (I.e. there is no characteristic length scale)

One RG iteration = 3 steps

Step I: Coarse grain.

Change resolution from $a$ to $2a$.

Integrate short distance fluctuations.

E.g. Summing over every other spin changes the lattice constant from $a$ to $2a$. 
(In real space, one would have \( l = \text{integer} \). In Fourier space \( l \) could be arbitrary.)

**Step II**

Rescale: \( x' = \frac{x}{l} \)

\[ \downarrow \text{Step I} \]

- I.e. measure lengths in units of the new lattice constant.
- Now the new lattice looks the same as the old one.

**Step III**

Spin rescaling / wave function renormalization

(to be considered later)
• As we iterate, we are successively looking at longer and longer distance physics.

• The relevant coupling constants grow. The irrelevant ones shrink.

• Assume we have done enough RG that all irrelevant coupling constants have already disappeared.

• Then the functional & form of \( H \) does not change, only its coefficients.

\[
H(t, h) \rightarrow H' = H(t', h')
\]

(assuming \( t, h \) are the only relevant parameters)

\[
Z = \text{Tr} \ e^{-H}
\]
• Fixed point \( t = t^*, h = h^* \) where
  \[ H'(t^*, h^*) = H(t^*, h^*) \]

• The system exhibits self similarity / scale independence
  at the fixed point

• Fixed point is not the same as critical point.

Different magnets at the critical point have different \( H \). However under RG, they flow to the same fixed point.
  \[ \Rightarrow \text{Universality!} \]

They all look identical at long distances.
• Behavior of $\xi$

\[ \dot{\xi} = \frac{\xi}{\ell} \]

(\(\xi\) is of course the same in laboratory units.)

→ Away from critical point

\[ \xi \to 0 \]

→ At fixed point \( \xi^* = \xi^* \Rightarrow \xi^* = \xi^* \)

\[ \Rightarrow \xi = 0 \text{ or } \xi = \infty \]

trivial fixed point non-trivial fixed point

• Main idea:

Linearize the RG flow equations in the vicinity of the fixed point.
For Ising model

\[ t^* = h^* = 0 \]

Flow equation:

\[ t' = f(t, h) \]
\[ h' = g(t, h) \]

Linearize:

\[ t' = A t + B h \]
\[ h' = C t + D h \]

Due to symmetry under \( h \rightarrow -h \), we have

\[ t' = A t \]
\[ h' = D h \]

(Otherwise diagonalize

\[ \begin{pmatrix} t' \\ h' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} t \\ h \end{pmatrix} \]

\[ t' = A(l) t \]
\[ h' = D(l) h \]

Clearly \( A(l_1) A(l_2) = A(l_1 l_2) \)

Solution \( A(\ell) = \ell \)

\( (\text{Group}) \)

\( \star \) Actually a semigroup.

\( \star \) No inverse.
\[
\begin{align*}
    t' &= e^{y_t} t \\
    h' &= e^{y_h} h
\end{align*}
\]

These eqns tell us how \( t \) and \( h \) vary under coarse graining.

Since the new \( H \) has the same form as the old one, the free energy will have the same functional form.

- After one step
  \[
  \frac{N}{l} f_s(t', h') = N f_s(t, h)
  \]

- After \( n \) steps
  \[
  \frac{N}{l} f_s(t^{(n)}, h^{(n)}) = N f_s(t, h)
  \]

  \[
  -\frac{d}{dt} f_s(l, \frac{ny_t}{l}, \frac{ny_h}{l}) = N f_s(t, h)
  \]

  True for arbitrary \( n \).

Choose \( l \) such that \( l^T t^T = 1 \) or \( l = \frac{1}{t^T} \).

\[
\Rightarrow \quad f_s(t, h) = |t|^d f_s(t^y, \frac{h}{t^{y_h/y_t}})
\]
We have "derived" scaling from RGI:

\[ f_s(t, h) = |t| f \left( 1, \frac{h}{|t| y_t / y_h} \right) \]

\[ 2 - \alpha = \frac{d}{y_t}, \quad \Delta = \frac{y_h}{y_t}, \quad F(x) = f_s(1, x) \]

A detailed calculation will give \( y_h, y_t \), from which all exponents can be derived.
1D Ising model

First the exact soln

\[ Z_N(h,k) = \text{Tr} \exp\left[ k \sum_i S_i S_{i+1} + h \sum_i \sum_i \right] \]

\[ = \prod_{i=1}^N \prod_{S_i S_{i+1} T_{S_1 S_2} T_{S_2 S_3} \cdots T_{S_N S_{N+1}} \cdots} \]

\[ = \text{Tr} T^N = \text{Tr} \left( \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix} \right)^N \]

\[ = \lambda_+^N + \lambda_-^N = \lambda_+^N \left( 1 + \left( \frac{\lambda_-}{\lambda_+} \right)^N \right) \]

\[ \rightarrow \lambda_+ \]

\[ T_{S_1 S_2} = \exp \left[ \frac{h}{2} (S_1 + S_2) + k S_1 S_2 \right] \]

\[ T = \begin{bmatrix} e^{h+k} & -e^k \\ -e^{-k} & e^{-h+k} \end{bmatrix} \]

\[ \lambda_+ \text{ and } \lambda_- \text{ are its eigenvalues} \]

\[ \lambda_{\pm} = e^{\frac{h}{2}} \left( \cosh h \pm \sqrt{\sinh^2 h + e^{4k}} \right) \]
\[ Z_N(h,k) = \lambda_{+}^{N} = e^{-\frac{F}{T}} \]

\[ f = \frac{F}{N} = -T \ln \lambda_{+} \]

\[ = -KT - T \ln (\cosh h + \sqrt{\sinh^2 h + e^{-4k}}) \]

\[ K = \frac{3}{T} \]

\[ J'' = \text{cost. drop it.} \]

\[ f = -T \ln (\cosh h + \sqrt{\sinh^2 h + e^{-4k}}) \]

\[ \text{exact!} \]

- \( f \) is analytic for any finite \( T \).
  \( \Rightarrow \) No phase transition at \( T \neq 0 \).

- \( \bar{M} = \left\langle s \right\rangle = -\frac{\partial f}{\partial H} = -\frac{\partial f}{T \partial h} \]

\[ = \frac{\sinh h}{\sqrt{\sinh^2 h + e^{-4k}}} \]

\[ h \to 0 \quad \Rightarrow \quad 0. \quad \text{No spontaneous symmetry breaking} \]
magnetic susceptibility

\[ \chi = \left. \frac{\partial m}{\partial H} \right|_{H \to 0} = \frac{\hbar}{e^2} \kappa = e^{\frac{2J}{T}} \frac{\hbar}{T} \]

\[ = \frac{\partial}{\partial H} \left( e^{\frac{2J}{T}} \frac{H}{T} \right) \]

\[ = \frac{1}{T} e^{\frac{2J}{T}} \]

\( \sim \frac{1}{T} \) for large \( T \) (Curie)

essential singularity at

\( \lim_{T \to 0} \) as \( T \to 0 \)

Heat capacity \( (H=0) \)

\[ C_V = -T \left. \frac{\partial^2 f}{\partial T^2} \right|_{H=0} = -2J/T \quad C_V \]

\[ \sim -T^4 \frac{4J}{T^3} \frac{e^{-2J/T}}{1 + e^{-2J/T}} \]

\[ \sim \frac{4J}{T^2} \exp \left( -\frac{2J}{T} \right) \]

has a peak near \( J \sim k_B T \)

(Schottky anomaly)

NO singularity as \( T \to 0 \)

(exponential dominates)
From

\[ \langle S_0 S_n \rangle \xrightarrow{T \to 0} -2n e^{-5/T} \]

\[ = e^{-\frac{n}{\xi}} \]

\[ = e^{\xi} \]

\[ \Rightarrow \xi = \frac{1}{2} \exp \left[ \frac{5}{T} \right] \]

- diverges as \( T \to 0 \)
- not power law but essential singularity
RG treatment of 1D Ising model

\[ H = -k \sum_j S_j S_{j+1} - h \sum_j S_j - C \geq 1 \]

\[ = H(k, h, C) \]

has no physical significance

\[ Z_N = \text{Tr} \ e^{-H} = \sum_{S_i=\pm 1} \exp[-H(S_i)] \]

"Divide and conquer."
Trace over every other spin.

Step I

\[ Z_N = \sum_{S_i} \cdots T_{S_- S_o} T_{S_o S_+} \cdots \]

Define

\[ T'_{S_- S_+} = \sum_{S_0=\pm 1} T_{S_- S_0} T_{S_0 S_+} \]

\[ = \exp[k' S_- S_+ + \frac{1}{2} h' (S_- S_+ + S_0) + C'] \]

(Recall \[ T_{S_- S_0} = \exp[k S_- S_0 + \frac{1}{2} h (S_- + S_0) + C] \])
\[
T'_{S_-S_+} = \left\{ \begin{array}{c}
\text{at } S_o = \pm 1 \\
T_{S_-S_o} T_{S_oS_+}
\end{array} \right.
\]

\[
= \exp \left( \left\{ K S_+ + \frac{1}{2} h S_+ + \frac{1}{2} h + c \right\} + \left\{ K S_+ + \frac{1}{2} h + \frac{1}{2} h S_+ + c \right\} \right)
+ \exp \left( -K S_- + \frac{1}{2} h S_- - \frac{1}{2} h + c - K S_- - \frac{1}{2} h + \frac{1}{2} h S_+ + c \right)
\]

\[
= \exp \left( \frac{1}{2} \left( S_+ + S_+ \right) + 2c \right) 2 \cosh \left[ K (S_++S_+) + h \right]
\]

\[
= \exp \left[ K' S_-S_+ + \frac{1}{2} h' (S_+ + S_+) + c' \right]
\]

We now solve for \( K', h', c' \) in terms of \( K, h, c \).

(i) \( S_- = S_+ = 1 \)

\[
e^{K' + h' + c'} = e^{h + 2c} \quad \Rightarrow \quad 2 \cosh \left( 2K + h \right) \quad (1)
\]

\( x = \)

(ii) \( S_- = S_+ = -1 \)

\[
e^{K' - h' + c'} = e^{-h + 2c} \quad \Rightarrow \quad 2 \cosh \left( -2K + h \right) \quad (2)
\]

\( y = \)

(iii) \( S_- = -S_+ = 1 \)

\[
e^{-K' + c'} = e^{2c} \quad \Rightarrow \quad 2 \cosh \left( h \right) \quad (3)
\]

3 equations, 3 unknowns.
Define \( e^k' = A \), \( e^h' = B \), \( e^c' = D \)

\[
\begin{align*}
A & = x \\
B & = y \\
D & = z
\end{align*}
\]

\( 1 \times \frac{2}{3} \times z^2 = ABD \)

\( \frac{AD}{B} \frac{D^2}{A^2} = D^4 = xy^2 \)

\[
\Rightarrow \quad e^{4c'} = e^8 \cdot 16 \cdot \cosh(2k+h) \cdot \cosh(-2k+h) \cdot \cosh^2 h
\]

\( A = \frac{D}{z} \)

\[
\Rightarrow \quad e^k' = \frac{e^c'}{e^{2c} \cdot 2 \cosh(h)}
\]

\[
\begin{align*}
4k' & = \frac{\cosh(2k+h) \cosh(-2k+h)}{\cosh^2 h} \\
B & = \frac{AD}{y}
\end{align*}
\]

\[
2h' = e^{2h} \frac{\cosh(2k+h)}{\cosh(2k-h)}
\]

-algebra 

- exact recursion relations/flow eq'ns for 1D Ising model
Fixed points

Put \( H = 0 \).
Flow of \( C \) is not relevant.

\[
e^{4K'} = \cosh^2(2K) = \frac{e^{4K} + e^{-4K}}{2}
\]

(\( \star \))

For fixed point, \( k' = k \).

(i) \( k = \infty \quad (\infty T = 0) \)

Then \( 4K' = \ln \frac{e^{4K}}{4} = 4K - \ln 4 \approx 4K \)

\( \Rightarrow k' \approx k \).

(ii) \( k = 0 \quad (T = \infty) \)

\[ e^{4K'} = 1 \Rightarrow k' = 0. \]

\( \text{RG flow?} \)

For large \( K \), \( e^{4K'} \approx e^{4K} \)

\( \Rightarrow k' \approx K \)

For small \( K \), from (\( \star \))

\[ 1 + 4K' \approx 1 + 4K^2 \]

\( \Rightarrow k' \approx K^2 \Rightarrow k' < K \)

\[ \begin{array}{ccccccccccccccc}
\infty & \rightarrow & \rightarrow & \rightarrow & 0 & \rightarrow & \rightarrow & \rightarrow & K \\
& \rightarrow & \rightarrow & \rightarrow & 0 & \rightarrow & \rightarrow & \rightarrow & 1
\end{array} \]
RG for 1D Ising model

0 \rightarrow \rightarrow \rightarrow \rightarrow T

(We are showing this as a continuous flow for clarity.)

- It is clear that $\xi = \infty$ at $T=0$ and $\xi = 0$ at $T=\infty$.

- $T=0$ is the critical/nontrivial fixed point.

- Notice $T=0$ is a fixed point. => There is a "phase transition" with $T_c = 0$.

- Phase diagram with both $h$ and $T$? Home assignment.
Free energy?

For one step

\[ \xi' = \frac{\xi}{2} = \frac{\xi}{x} \]

\[ \ln Z_N(t, h) = \ln Z_{N'}(t', h') \]

\[ \frac{1}{N} \ln Z_N(t, h) = \frac{1}{N'} \ln Z_{N'}(t', h') \]

\[ f(t, h) = \frac{1}{x} f(t', h') \]

After \( n \) steps

\[ f(t, h) = \frac{1}{L^n} f(t^{(n)}, h^{(n)}) \]

Put \( h = 0 \).

Convenient to work with \( x = e^{-\frac{4K}{T}} \)

\[ e^k = \frac{\cosh 2K}{\sinh 2K} = \frac{e^k + e^{-k}}{2} \]

For \( T \to 0 \)

\[ x' = 4 \frac{x}{k^2} = \frac{x}{k^2} \]

\[ f(x) = \frac{1}{L^n} f \left( \frac{x L^{2n}}{k^{2n}} \right) \]

Define \( x L^{2n} = 1 \) \( \Rightarrow \) \( \frac{1}{k^2} = x^{-2n} \)

\[ f(x) = x^{-2K} f(x^{1}) = e^{-2K} f(1) \]
\[ f(T) = e^{-\frac{2J}{t}} f(1) \]

Compare with the exact result \( f(T) = -T \ e^{-\frac{2J}{t}} \).

At least we get the essential singularity correctly.

Imagine a model where

\[ T' = R(T) \]
RG generates new couplings - the problem appears to become more complicated in the beginning!

\[
H = J_1 \sum_i (S_i \cdot S_{i+1} + \sigma_i \cdot \sigma_{i+1}) + J_2 \sum_i S_i \cdot \sigma_i \cdot \sigma_{i+1}
\]

integrate out half of spins (boxes)

\[
H = J_1' \sum_i (S_i \cdot S_{i+1} + \sigma_i \cdot \sigma_{i+1}) + J_2' \sum_i S_i \cdot \sigma_i \cdot \sigma_{i+1}
\]

- New couplings \((J_3', J_4')\) are generated. These "happened" to be zero initially.

- In general one must keep all couplings in \(H\) for useful RG.

- Not many RG problems solved exactly.

- After enough RG, many couplings shrink to zero, however.