Chapter 8

Thermodynamic Formalism of Expanding and Hyperbolic Maps

Definition 1: A d-dimensional map $f$ on the phase plane $\mathbb{R}^2$ is called expanding if there exist constants $\varepsilon > 0$ and $\alpha > 1$ such that

$$||f(x) - f(y)|| > \alpha ||x - y||$$

for all $x, y \in \mathbb{R}$ such that $||x - y|| < \varepsilon$.

Locally, each point in the phase plane expands and remains distinct.

Definition 2: A d-dimensional map $f$ on the phase plane $\mathbb{R}$ is called hyperbolic if the stable and unstable manifolds of the unstable fixed points always cross each other transversely; they must not touch each other.

Remark: For a linear (or linearized) hyperbolic map, no eigenvalue of the Jacobian must lie on the unit circle.

Variational Principle for Topological Pressure

Let us consider a d-dimensional expanding map $x_{n+1} = f(x_n)$ and each N-cylinder be labeled by an index $j$. We use a variational principle to find the invariant density of expanding maps similar to what was done for free energy.

Definition 3: Topological pressure wrt a test function $\varphi(x)$ is defined as:

$$\varphi(\varphi) = \lim_{N \to \infty} \frac{1}{N} \ln \left( \sum_{j=1}^{k(N)} \exp \left( S_N \varphi(x_j) \right) \right)$$

(1)

where

$$S_N \varphi(x) = \varphi(x) + \varphi(f(x)) + \ldots + \varphi(f^{N-1}(x))$$

(2)

The point $x_j$ is located in the $j$th N-cylinder.

The local expansion rate is defined as:

$$E_j^{(N)} := E_N(x_j) := -\beta^{-1} S_N \varphi(x_j)$$

(3)

Remark: $E_j^{(N)}$ is interpreted as the energy of the $j$th cylinder with escort probability

$$P_N(x_j) = \frac{1}{Z_N^{top}(\beta)} \exp \left[ -\beta N E_N(x_j) \right]$$

(4)

where

$$Z_N^{top}(\beta) = \sum_j \exp \left( -\beta N E_j^{(N)} \right)$$

(5)

The topological pressure is given as:

$$\varphi(\beta) = \lim_{N \to \infty} \frac{1}{N} \ln Z_N^{top}(\beta)$$

(6)
Remark: \( \mathcal{P}(\mathcal{F}(\beta)) \) is equivalent to the time-averaged free energy \( \Psi \) of all \( N \)-cylinders as \( N \to \infty \). Hence, the principle of minimum free energy becomes the principle of maximum topological pressure.

Let \( p_j^{(N)} \) be the probability of \( j \)-th \( N \)-cylinder. Then,

\[
\Psi_N(\mathcal{F}(\cdot), \beta) = \sum_j \left( p_j^{(N)} \beta E_j^{(N)} + p_j^{(N)} \ln p_j^{(N)} \right)
\]

(7)

Recall

\[
\Psi(\beta) = \beta \sum_i E_i^0 - \frac{S}{\beta} = \sum_{i=1}^T \left( \beta_i E_i^0 - p_i \ln p_i \right)
\]

because \( \sum_{i=1}^T p_i = 1 \)

The escort probability at a thermodynamic equilibrium (i.e., minimum energy)

\[
P_j^{(N)} = \exp \left[ \frac{\Psi_N + S_N \mathcal{F}(\chi_0^{(j)})}{\beta} \right]
\]

(8)

\[\Rightarrow\] free energy \( = -\ln \left( \sum_j \exp \left[ S_N \mathcal{F}(\chi_0^{(j)}) \right] \right) = \min_{\mathcal{F}} \Psi_N(\mathcal{F}(\cdot), \beta) = \Psi_N(\mathcal{F}(\cdot)) \)

(9)

Eqs. (7) and (3) yield the minimum of time-averaged quantities as \( N \to \infty \):

\[
\lim_{N \to \infty} \left( -\frac{1}{N} \Psi_N(\mathcal{F}(\cdot), \beta) \right) = \lim_{N \to \infty} \left( \frac{1}{N} \sum_j p_j^{(N)} S_N(\mathcal{F}(\chi_0^{(j)})) - \frac{1}{N} \sum_j p_j^{(N)} \ln p_j^{(N)} \right)
\]

(10)

We observe in (1) and (9) that the left side of (10) converges to the topological pressure by taking supremum over all possible probability distributions \( \mu \):

\[
\mathcal{P}(\mathcal{F}(\cdot)) = \sup_{\mu} \lim_{N \to \infty} \left[ -\frac{1}{N} \Psi_N(\mathcal{F}(\cdot), \beta) \right]
\]

(11)

Where the measure \( \mu \) yields

\[
p_j^{(N)} = \int_{J_j^{(N)}} \text{d} \mu(x)
\]

(12)

Remark: The supremum in (11) is NOT necessary for a generating partition.
The first term on the right side of (10) is now considered under invariant measures \( \mu \) of the map \( f \) \( [\text{Note: } x_{k+1} = f(x_k)] \). Then, for a test function \( g(x) \) with \( N \to \infty \), we have

\[
\sum_j b_j^{(N)} g(f(x_0^{(j)})) = \sum_j b_j^{(N)} g(x_0^{(j)}) \quad \text{because of the invariance property}
\]

\[
\Rightarrow \frac{1}{N} \sum_j b_j^{(N)} S_N(\varphi(x_0^{(j)})) = \sum_j b_j^{(N)} \varphi(x_0^{(j)})
\]

Now,

\[
\lim_{N \to \infty} \frac{1}{N} \sum_j b_j^{(N)} S_N(\varphi(x_0^{(j)})) = \lim_{N \to \infty} \frac{1}{N} \sum_j b_j^{(N)} \varphi(x_0^{(j)})
\]

\[
= \langle \varphi \rangle \quad \text{because of ergodicity} \quad (14)
\]

The second term on the right side of (10) yields

\[
\lim_{N \to \infty} \left( -\frac{1}{N} \sum_j b_j^{(N)} \ln b_j^{(N)} \right) = h(\mu) \quad \text{entropy rate} \quad (15)
\]

Now, we express topological pressure from (11) in (10) as:

\[
P(\varphi(\cdot)) = \sup_{\mu} \left( \langle \varphi \rangle + h(\mu) \right) \quad (16)
\]

Remark:

Topological pressure \( P \sim \) free energy that takes the minimum value under thermodynamic equilibrium

\( h(\mu) \sim \) entropy rate

\( \langle \varphi \rangle \) mean of the test function \( \sim \) mean energy

Remark: the above approach requires existence of proper \( N \)-cylinders. The topological pressure \( P(\varphi(\cdot)) \) should be independent of the choice of the initial conditions \( x_0^{(j)} \).
Length-Scale Interpretation

Let us assume that there exists a generating partition of the phase space that consists of a finite number of cells $i$ of size $L_i$.

An $N$-cylinder $J(i_0, i_1, \ldots, i_{N-1})$ is interpreted as the set of initial points that generate the symbol sequence $i_0 i_1 \cdots i_{N-1}$. After $N$ iterations from the cell $i_0$, the trajectory reaches the cell $i_{N-1}$. For a given $N$, a specific $N$-cylinder is labelled as $J^{(k)}_N$. For example, $J^{(1)}_1$ is a cell in the phase space. In one-dimensional case, the $J^{(k)}_1$'s are intervals.

In case of a generating partition, each $N$-cylinder shrinks to a very small interval for a sufficiently large $N$; this is based on the local expansion rate. Let the length of the interval onto which the cylinder $J^{(k)}_N$ is mapped be denoted as $l^{(k)}$. The index $i$ is a function of the cylinder index $k$. Then, the interval length $l^{(k)}_{N+1}$ within which the initial condition $x_0^{(k)}$ lies after $N$ iterations is given as:

$$l^{(k)}_{N+1} = l^{(k)} \exp (-N \varepsilon^{(k)}_N)$$

(17)

where $\varepsilon^{(k)}_N = E_N(x_0^{(k)}) = \frac{1}{N} \sum_{n=0}^{N-1} \ln |f'(x_n)|$ for the dynamical system $x_{n+1} = f(x_n)$.

The topological partition function is expressed as:

$$Z_N^{\text{top}}(\beta) = \sum_j \left( \frac{l^{(j)}_{N+1}}{\varepsilon^{(j)}_N} \right)^\beta$$

(18)

Note that since $l^{(j)}_{N+1}$ is bounded, the smallest $c_{\text{min}}$ and largest $c_{\text{max}}$ size cells of the generating partition, we have, for $\beta > 0$,

$$c_{\text{max}}^{-\beta} \sum_j \left( \frac{l^{(j)}_{N+1}}{\varepsilon^{(j)}_N} \right)^\beta \leq Z_N^{\text{top}}(\beta) \leq c_{\text{min}}^{-\beta} \sum_j \left( \frac{l^{(j)}_{N+1}}{\varepsilon^{(j)}_N} \right)^\beta$$

(19)

$$\Rightarrow -\frac{\beta}{N+1} \ln c_{\text{max}} + \frac{1}{N+1} \ln \left( \sum_j \left( \frac{l^{(j)}_{N+1}}{\varepsilon^{(j)}_N} \right)^\beta \right) \leq \frac{\ln Z_N^{\text{top}}(\beta)}{N+1} \leq -\frac{\beta}{N+1} \ln c_{\text{min}} + \frac{1}{N+1} \sum_j \varepsilon^{(j)}_N \beta$$

As $N \to \infty$, 

$$\lim_{N \to \infty} \frac{1}{N} \ln \sum_j \left( \frac{l^{(j)}_{N+1}}{\varepsilon^{(j)}_N} \right)^\beta = \mathcal{O}(\beta)$$

(20)

Similar expressions can be derived for $\beta < 0$. 
**Gibbs Measure**

For every test function $\Phi$, there is an invariant measure $\mu_\Phi$, called the Gibbs measure, that maximizes the topological pressure $P(\Phi) = \sup_{\mu} \left( \langle \Phi \rangle + h(\mu) \right)$

The corresponding canonical distribution (see (8) and (11))

$$P_j^{(N)} = \mu\left( J_j^{(N)} \right) = c_j^{(N)} \exp \left[ -N P(\Phi) + S_N \Phi(x_0^{(j)}) \right]$$

where the constants $c_j^{(N)}$ are bounded between minimum and maximum cell sizes, $c_{\text{min}}$ and $c_{\text{max}}$. However, as $N \to \infty$, their effects may become vanishingly insignificant.

Let us now focus on the one-dimensional map $x_{n+1} = f(x_n)$.

Let the test function $\Phi(x)$ be chosen as:

$$\Phi(x) = -\beta \ln |f'(x)|$$

For each $\beta$, there is an invariant measure $\mu_\beta$, called the Gibbs measure, that minimizes the free energy. For $\beta = 1$, Gibbs measure $\mu_1$ is called the Sinai-Ruelle-Bowen (SRB) measure.

We show next that the SRB measure is identical to the natural invariant measure. [Recall that there is only one invariant measure for an ergodic map, out of possibly many invariant measures, in the sense that the map will converge almost surely if iterated infinitely from randomly chosen initial points; this measure is called the natural invariant measure.]

$$P_j^{(N)} = \exp \left[ S_N \Phi(x_0^{(j)}) \right] \sum_k \exp \left[ S_N \Phi(x_0^{(k)}) \right] = \left( \frac{c_j^{(N)}}{c_k} \right)^\beta \sum_k \left( \frac{c_k^{(N)}}{c_k} \right)^\beta \quad \text{[see Appendix in p.10]}$$

Note that, from (17), we have $l_j^{(N)} = L_{1(j)}^\beta \exp \left[ -N E_j^{(N)} \right]$ and $E_j^{(N)} = E_N(x_0^{(j)}) = -\beta^{-1} S_N \Phi(x_0^{(j)})$ from (3). Combining these two expressions, we have

$$\exp \left[ -N S_N \Phi(x_0^{(j)}) \right] = \left( \frac{c_j^{(N)}}{c_k} \right)^\beta$$

Remark: Gibbs measure $\mu_\beta$ attributes to each cylinder a probability proportional to the cylinder size $l_j^{(N)}$ raised to the power $\beta$. 
Remark: In contrast to the conventional escort distribution $R_i \equiv \exp(i - \beta E_i)$, the escort distribution in (22) is derived w.r.t. cylinder lengths, rather than probabilities of boxes of equal size. For $\beta = 1$, the SRB measure $\mu$ is a smooth measure because all probabilities are proportional to (bounded) cylinder lengths.

Next let us focus on the bounding constants $C_j^{(n)}$ in (21).

Definition 4: Let the natural invariant measure $\mu$ for the symbol sequences of a map, $x_{n+1} = f(x_n)$, be given as: $\mu(i_0, \ldots, i_{n-1}) = \mu(J[i_0, \ldots, i_{n-1}])$. Then, the stochastic symbolic process $x^n$ is defined to be a Markov chain if

$$p(i_n | i_{n-1}, \ldots, i_0) = p(i_n | i_{n-1}) \forall n \in \mathbb{N}$$

and is defined to be a topological Markov chain if it has the following property:

$$\left( p(i_n | i_{n-1}, \ldots, i_0) = 0 \right) \text{ if and only if } \left( p(i_n | i_{n-1}) = 0 \text{ or } p(i_{n-1} | i_{n-2}, \ldots, i_0) = 0 \right) \quad (23)$$

Remark: The implication of the topological Markov property is as follows:

- The transition probability $p(i_n | i_{n-1}, \ldots, i_0)$ to cell $i_n$ is 0 if either it is not possible to reach $i_n$ from $i_{n-1}$, or the sequence $i_0 \ldots i_{n-1}$ is forbidden.

Remark: In general, stochastic sequences, generated by a map, is neither Markov nor topologically Markov because they are characterized by the chosen partition functions. For expanding maps dealt with in this chapter, it is possible to have a stochastic symbolic process to be topologically Markov, such a partition is called Markov partition. An essential property of a Markov partition is that an arbitrary $N$-cylinder can be expressed as the union of a specific set of $(N+1)$-cylinders, i.e.,

$$J_j^{(n)} = \bigcup_k J_{k_l}^{(N+1)} \quad (24)$$

Consequently, because of pairwise disjoint $J_j^{(N+1)}$'s, the invariant measure is:

$$\mu(J_j^{(n)}) = \sum_k \mu(J_{k_l}^{(N+1)}) \quad (25)$$
An application of (25) in (21) yields
\[ C_j^{(N)} \exp[-Np(\omega)] \exp[\mathcal{S}_N \varphi(x_0^{(j)})] = \prod_{k=1}^N C_k \exp[-(N+1)p(\omega)] \exp[\mathcal{S}_{N+1} \varphi(x_0^{(k)})] \]
\[ \Rightarrow C_j^{(N)} \exp[\mathcal{S}_N \varphi(x_0^{(j)})] = \exp[-\beta \sum_{n=0}^{N-1} \ln |f'(x_n^{(j)})|] = \left(\frac{f_{N+1}}{f_N}\right)(x_0^{(j)})^{-\beta} \]
\[ \exp[\mathcal{S}_{N+1} \varphi(x_0^{(k)})] = \left| \left(\frac{f_{N+1}}{f_N}\right)(x_0^{(k)}) \right|^{-\beta} \]

Considering the test function \( \varphi(x) = -\beta \ln |f'(x)| \), we have

\[ \exp[\mathcal{S}_N \varphi(x_0^{(j)})] = \exp[-\beta \sum_{n=0}^{N-1} \ln |f'(x_n^{(j)})|] = \left(\frac{f_{N+1}}{f_N}\right)(x_0^{(j)})^{-\beta} \]
\[ \text{and} \quad \exp[\mathcal{S}_{N+1} \varphi(x_0^{(k)})] = \left| \left(\frac{f_{N+1}}{f_N}\right)(x_0^{(k)}) \right|^{-\beta} \]

For a generating partition, as \( N \to \infty \), \( J_j^{(N)} \) converges to a point set for all \( j \). So, \( x_0^{(j)} \) is the inverse image of \( x_j^{(j)} \) under \( f \), i.e., \( f(x_0^{(j)}) = y \).

For limiting values of the constants \( C_j^{(N)} \), we set
\[ \lim_{N \to \infty} C_j^{(N)} = P(y) \text{ and } \lim_{N \to \infty} C_j^{(N)} = P(x_0) \]

As \( N \to \infty \), a combination of (26) to (29) yields the following limit:
\[ P(y) \left| \left(\frac{f_{N+1}}{f_N}\right)(y) \right|^{-\beta} = \exp\left[-\beta \sum_{k=1}^{2} P(x_k) \left| \left(\frac{f_{N+1}}{f_N}\right)(x_k) \right|^{-\beta} \right] \]

Further, since \( \left(\frac{f_{N+1}}{f_N}\right)(x_k) = \frac{df}{d\theta} \bigg|_{\theta=x_k} \frac{df}{d\theta} \bigg|_{\theta=f^{-1}(x_k)} = f'(x_k)f_{N+1}'(y) \),

it follows that
\[ P(y) = \exp\left[-\beta \sum_{k=1}^{2} P(x_k) \left| f'(x_k) \right|^{-\beta} \right] \]

The invariant density \( P(y) \) is an eigenfunction of the Perron-Frobenius operator for \( \beta = 1 \). If \( \beta \to 1 \), i.e., if \( \beta \to 0 \), it escapes rate \( \epsilon = 0 \).

**Example:** Consider the binary shift map \( f(x) = \begin{cases} 2x & \text{for } x \in [0, \frac{1}{2}) \\ 2x - 1 & \text{for } x \in [\frac{1}{2}, 1) \end{cases} \)

For SRB measure (i.e., Gibbs measure with \( \beta = 1 \)),

the factors \( C_j^{(N)} \) are equal to 1 because \( P(y) = 1 \) a.e. on \([0, 1]\).

For a non-expanding map like Ulam map \( f(x) = 1 - 2x^2 \) (because \( \left| f'(x) \right| = 0 \) at \( x = 0 \)), we have \( P(x) = \frac{1}{\sqrt{1-x^2}} \) and the invariant measure \( \mu(J_j^{(N)}) = \int dx P(x) \) is not an SRB measure because \( J_j^{(N)} \) is not proportional to the length of cylinders everywhere. Note that Ulam map is not an expanding map.
Relation between Topological Pressure and Renyi Entropy

We have seen that, for one-dimensional expanding maps, the SRB measure yields the natural invariant density. Now, we show that the topological pressure determines dynamical Renyi entropies.

Recall that dynamical Renyi entropy of order $\beta$ is given by

$$K(\beta) = \frac{1}{\beta} \log \mu([A])$$

where $\{A\}$ is a partitioning and $\mu$ is the natural invariant density.

$$h_{\beta}(A) = \lim_{N \to \infty} \frac{1}{N} \log Z_{N}^{\text{top}}(\beta)$$

$$= \lim_{N \to \infty} \frac{1}{N} \log \prod_{j=1}^{N} (P_{j}^{(N)})^\beta$$

We generalize the dynamical Renyi entropy by arbitrary Gibbs measure $\mu_{q}$ parametrized by $\frac{1}{q}$. These are denoted by $K(\beta, q)$ where $K(\beta, 1) = K(\beta)$. Then, we define

$$P_{j}^{(N)} \sim \sum_{j} \left( \frac{f_{j}(N)}{Z_{N}^{\text{top}}(q)} \right)^{\beta}$$

Earlier, we defined $P_{j}^{(N)} \sim \left( \frac{f_{j}(N)}{Z_{N}^{\text{top}}(q)} \right)^{\beta}$ where $Z_{N}^{\text{top}}(q) = \sum_{j} \left( \frac{f_{j}(N)}{Z_{N}^{\text{top}}(q)} \right)^{\beta}$

Topological pressure $P(q) = \lim_{N \to \infty} \frac{1}{N} \log Z_{N}^{\text{top}}(q) \Rightarrow Z_{N}^{\text{top}}(q) \sim \exp[NP(q)]$

Hence, $\sum_{j} \left( \frac{f_{j}(N)}{Z_{N}^{\text{top}}(q)} \right)^{\beta} \sim \exp[-NP(q)] \sum_{j} \left( \frac{f_{j}(N)}{Z_{N}^{\text{top}}(q)} \right)^{\beta}$

From (31), $\sum_{j} \left( \frac{f_{j}(N)}{Z_{N}^{\text{top}}(q)} \right)^{\beta} \sim \exp[-NP(q)] \exp[NP(q)]$

$$\Rightarrow K(\beta, q) = \frac{1}{(1-\beta)} (P(q) - \beta P(q))$$

Denoting the escape rate $K = -P(1)$, we have for $q = 1$,

$$K(\beta) = K(\beta, 1) = \frac{1}{1-\beta} [P(\beta) + KP]$$
Relations between Topological Pressure and Generalized Lyapunov Exponents

We have seen that, for a one-dimensional expanding map, a small interval containing an initial value \( x_0 \) expands after \( N \) iterations by a factor

\[
| (f^N)'(x_0) | = \exp (N \lambda_N(x_0))
\]

where \( \lambda_N(x_0) = \frac{1}{N} \sum_{j=0}^{N-1} \ln | f'(x_j) | \) is the local Lyapunov exponent at \( x_0 \)

\[
\left\langle | (f^N)'(x_0) | \right\rangle = \int d\mu(x_0) | (f^N)'(x_0) | = \int d\mu(x_0) \exp [N \lambda_N(x_0)]
\]

Define a partition function

\[
Z_N^L(\beta) = \left\langle | (f^N)'(x_0) |^\beta \right\rangle = \int d\mu(x_0) \exp [\beta N \lambda_N(x_0)]
\]

Define free energy per particle

\[
\Delta(\beta,N) = \frac{1}{\beta N} \ln Z_N^L(\beta)
\]

and the generalized Lyapunov exponent

\[
\Lambda(\beta) = \lim_{N \to \infty} \frac{1}{\beta N} \ln \left( \int d\mu(x_0) \exp \left( \beta \sum_{j=0}^{N-1} \ln | f'(x_j) | \right) \right)
\]

For a set of \( N \)-cylinders \( J_j^{(N)} \), the above integral is discretized as

\[
Z_N^L(\beta) \approx \sum_j P_j^{(N)} \exp \left( \beta N \sum_j E_j^{(N)}(\beta) \right) \sim \sum_j P_j^{(N)} (l_j^{(N)})^{-\beta}
\]

where \( P_j^{(N)} \equiv \int_{J_j^{(N)}} d\mu(x_0) \) and \( E_j^{(N)} = E_N(x_j^{(N)}) = \frac{1}{N} \sum_{n=0}^{N-1} \ln | f'(x_n) | \)

The Gibbs measure of order \( \beta \) yields

\[
P_j^{(N)} \sim (l_j^{(N)})^{\beta} \exp \left[ -N \mathcal{P}(\beta) \right] \quad \text{(See (32) and (33))}
\]

and we have

\[
Z_N^L(\beta) \sim \sum (l_j^{(N)})^{\beta} \exp \left[ -N \mathcal{P}(\beta) \right]
\]
Since \( \sum_j (\hat{p}_j^{(N)})^{q-\beta} \sim \exp \left[ N \varphi(q, -\beta) \right] \) for large \( N \), we have

\[
Z_N^L(\beta) \sim \exp \left[ N \varphi(q, -\beta) - \varphi(q) \right] \tag{41}
\]

Defining a generalized Liapunov exponent with respect to Gibbs measure of order \( q \) as:

\[
\Lambda(\beta, q) = \lim_{N \to \infty} \frac{1}{N} \ln Z_N^L(\beta) \tag{42}
\]

we have

\[
Z_N^L(\beta) \sim \exp \left[ N \Lambda(\beta, q) \right] \tag{43}
\]

A comparison of (41) and (43) yields

\[
\Lambda(\beta, q) = \beta^{-1} \left( \varphi(q, -\beta) - \varphi(q) \right) \tag{44}
\]

For \( q = 1 \), (44) yields

\[
\Lambda(\beta) = \Lambda(\beta, 1) = \beta^{-1} \left( \varphi(1, -\beta) + k \right)
\]

where \( k = -\varphi(1) \) is the escape rate.

Appendix

Topological Pressure for Arbitrary Test Functions

For an arbitrary test function \( \varphi(.) \), the topological pressure of a one-dimensional map is:

\[
P[\varphi] = \lim_{N \to \infty} \frac{1}{N} \sum_j \exp \left[ S_N \varphi(x_0^{(j)}) \right]
\]

where \( x_0^{(j)} \) denotes some point of the ensemble \( H_N(x_0, \ldots, x_{N-1}) \) which is the set of initial values \( x_0^{(k)} \), \( k = 1, \ldots, K(N) \) in \( \Gamma(N) \) cylinders.

Let \( \varphi(x) = -\beta \ln |f'(x)| \) where \( x_{n+1} = f(x_n) \) is the map.

Then, \( (S_N \varphi)(x) = \varphi(x) + (\varphi f)(x) + \cdots + (\varphi f^{N-1})(x) \) yields

\[
(S_N \varphi)(x_0) = -\beta \sum_{k=0}^{N-1} \ln |f'(x_k)|
\]

\[
\Rightarrow \exp [S_N \varphi(x_0^{(j)})] = \exp \left[ -\beta \sum_{k=0}^{N-1} \ln |f'(x_k^{(j)})| \right] = \exp \left[ -\beta \ln |f^{(N)}(x_0^{(j)})| \right] = |f^{(N)}(x_0^{(j)})|^{-\beta}
\]

\[
\Rightarrow P[\varphi] = \lim_{N \to \infty} \frac{1}{N} \sum_j |f^{(N)}(x_0^{(j)})|^{-\beta}
\]
Relationship between Topological Pressure and Renyi Dimension

Using cells of variable size, we have derived a notion of Renyi dimension that is analogous to Hausdorff dimension.

Let us cover the multifractal region of the phase space by disjoint cells \( \Sigma_1, \Sigma_2, \ldots, \Sigma_T \), where each cell may have different size and shape. Let \( p_i \) be the probability associated to cell \( \Sigma_i \). Let \( L_i \) be the diameter of the cell \( \Sigma_i \). Define a generalized partition:

\[
Z(\beta; t) = \lim_{\epsilon \to 0} \inf \left\{ \sum_{i=1}^{T} \left( \frac{p_i}{l_i} \right)^{t} : \sum_{i=1}^{T} \left( \frac{p_i}{l_i} \right)^{\beta} \leq 1 \right\}
\]

for \( \beta \leq 1 \) and \( t \leq 0 \),

\[
Z(\beta; t) = \lim_{\epsilon \to 0} \sup \left\{ \sum_{i=1}^{T} \left( \frac{p_i}{l_i} \right)^{t} : \sum_{i=1}^{T} \left( \frac{p_i}{l_i} \right)^{\beta} \geq 1 \right\}
\]

for \( \beta > 1 \) and \( t > 0 \). (45)

Recall how Hausdorff dimension is defined:

\[
\dim_H(-\Omega) = \begin{cases} 
\inf \{ t \in \mathbb{R} : H^t(-\Omega) = 0 \} & \text{if } 0 < \dim_H(-\Omega) < \infty \\
\sup \{ t \in \mathbb{R} : H^t(-\Omega) = \infty \} & \text{if } \dim_H(-\Omega) = \infty
\end{cases}
\]

So that \( H^t(-\Omega) = \begin{cases} 
\infty & \text{if } t < \dim_H(-\Omega) \\
0 & \text{if } t > \dim_H(-\Omega)
\end{cases} \)

If \( t = \dim_H(-\Omega) \), then \( 0 \leq H^t(-\Omega) \leq \infty \).

The Renyi dimension \( D \) is defined as:

\[
D(\beta) = \frac{t}{\beta - 1}
\]

where \( t \) is satisfied by (45), and \( Z(\beta; t) \) neither diverges or goes to 0 as \( T \to \infty \). (46)

Now, let us assume that we have a generating partition and a one-dimensional expanding map. Let the cell in \( N \)-cylinders \( J_j^{(N)} \), which has diameter \( l_j^{(N)} \) and probability \( p_j^{(N)} \), be assigned Gibbs measure \( P_j^{(N)} \) corresponding to the parameter \( \beta \). Note that, due to assumption of generating partition, the infimum, respectively supremum, has already been reached. For large \( N \),

\[
p_j^{(N)} \approx \left( \frac{l_j^{(N)}}{p_{j}^{(N)}} \right)^{\beta} \exp \left[ -N \Phi(\beta) \right]
\]

(47)

\[
\lambda_j^{(N)} \approx \left( \frac{p_{j}^{(N)}}{l_j^{(N)}} \right)^{\beta}
\]

(48)
For $p = 0$, the Bowen-Ruelle formula is:
\[ P[D(0)] = 0 \quad \text{(54)} \]

Remark: Hausdorff dimension is given by the value of $p$ for which the topological pressure $P(p)$ vanishes.

For $p = 1$, (53) reduces to the known result:
\[ K = -P(1) \]

Note: Let us cover the phase space $\Omega$ uniformly with a sufficiently large number of initial points. After $N$ iterations, let $m_N$ be the number of trajectories remaining in $\Omega$. For large $N$, one may expect an exponential decay of $m_N$, i.e.,
\[ m_N = m_0 \exp(-KN) \quad \Rightarrow \quad K = -\frac{1}{N} \ln \frac{m_0}{m_N} = -P(1) \]