Observer-Embedded $L_2$-Gain Control

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Abstract—This paper demonstrates that the $H_{\infty}$ controller of Doyle et al. [1] can be reinterpreted as an observer-embedded $L_2$-gain controller. The embedded observer is similar to the conventional Luenberger observer except having an additional state-dependent calibration term that compensates for exogenous inputs and model uncertainties. Based on this fact, we derive mixed Linear Matrix Inequalities, Hamiltonian Matrix, and Linear Parameterization to provide solutions to feasible observer-embedded $L_2$-gain controllers. © 2001 Elsevier Science Ltd. All rights reserved.

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INTRODUCTION

This paper presents synthesis of feasible observer-embedded $L_2$-gain controllers for the following generalized plant

$$
x = Ax + B_1w + B_2u,
$$

$$
z = C_1x + D_{11}w + D_{12}u,
$$

$$
y = C_2x + D_{21}w + D_{22}u,
$$

where the controller processes the measurable output signal $y$ to generate the control signal $u$ in presence of exogenous disturbances and modeling uncertainty $w$. For comparative study, the assumptions used here are the same as those of the DGKF $H_{\infty}$ controller [1] as follows

- $D_{11} = 0$, $C_1^TD_{12} = 0$, $D_{12}^TD_{12} = I$, $D_{21}B_1^T = 0$, $D_{21}D_{21}^T = I$, $D_{22} = 0$
- Both triples $(A, B_1, C_1)$ and $(A, B_2, C_2)$ are stabilizable and detectable

The objective of robust performance is specified as

$$
\int_0^T \|z\|^2 dt \leq \gamma^2 \int_0^T \|w\|^2 dt, \quad \forall T > 0, \quad \forall w \in L_2[0, T]
$$

under zero initial conditions. The scalar parameter $\gamma > 0$ signifies the specified performance related to energy gain of the closed-loop system.

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SYNTHESIS OF OBSERVER-EMBDEDDED $L_2$-GAIN CONTROL

The controller is postulated to have an embedded observer similar to the conventional Luenberger observer except having an additional state-dependent calibration term. The structure of the observer-embedded control law is delineated below.

$$\dot{x} = (A + W_{\text{cal}}) x + B_2 u + L (y - C_2 \dot{x}),$$
$$u = K \dot{x},$$

(3)

where $K$ is the controlling matrix, $L$ is the observing matrix, and $W_{\text{cal}}$ is the calibration matrix. Unlike $H_2$ optimal control, the resulted closed-loop system under the control structure (3) is calibrated to reduce sensitivity to exogenous inputs and modeling uncertainties $w$. This is achieved by introducing the calibration term $W_{\text{cal}}$ that is determined simultaneously with the controller matrix $K$ and the observer matrix $L$. The task is to synthesize a control law based on the generalized plant model (1) and the control objective (2) in two coupled stages of State-Feedback and State-Estimation.

In the state feedback stage, we choose a quadratic Lyapunov functional $V(x) = x^T X x$, $X = X^T > 0$ along the trajectory specified in the generalized plant in (1). The resulting dissipation rate is obtained as

$$V = -\|z\|^2 + \gamma^2 \|w\|^2 + x^T (A^T X + X A + C_1^T C_1 - F_2^T F_2 + \gamma^2 F_1^T F_1) x$$
$$- \gamma^2 \|w - F_1 x\|^2 + \|u - F_2 x\|^2$$

$$\equiv -\|z\|^2 + \gamma^2 \|w\|^2 + x^T Q_X x - \gamma^2 \|\dot{w}\|^2 + \|\ddot{u}\|^2,$$

(4)

where $F_1 \equiv \gamma^{-2} B_1^T X$, $F_2 \equiv -B_2^T X$, $\dot{w} \equiv w - F_1 x$, $\ddot{u} \equiv u - F_2 x$ and the first Riccati matrix $Q_X \equiv A^T X + X A + C_1^T C_1 - X B_2 B_2^T X + \gamma^{-2} X B_1 B_1^T X$

(5)

The controller gain matrix is chosen as $K = F_2$.

The dynamics of estimation error $\ddot{z} \equiv x - \dot{x}$ is governed by

$$\ddot{z} = x - \dot{x} = (A - Z C_2^T C_2) \ddot{z} + (B_1 - Z C_2^T D_{21}) \dot{w} + B_1 F_1 x - W_{\text{cal}} \dot{x}$$

This leads to derivation of the calibration matrix

$$W_{\text{cal}} = B_1 F_1$$

(6)

Consequently, the dynamics of estimation error becomes

$$\ddot{z} = (A + B_1 F_1 - Z C_2^T C_2) \ddot{z} + (B_1 - Z C_2^T D_{21}) \dot{w},$$

(7)

where the observer gain matrix has been set as $L = Z C_2^T$. The estimation accuracy decreases (increases) as the desired $\gamma$ becomes smaller (larger). The estimation accuracy is also increased if $(A - L C_2)$ has eigenvalues with large negative real parts.

In the state estimation stage, we choose the Lyapunov function $W = \ddot{z}^T Z^{-1} \ddot{z}$, $Z = Z^T > 0$ along the dynamical trajectory of the estimation error in (7). The resulting dissipation rate is obtained as

$$W = -\frac{1}{\gamma^2} \|\ddot{u}\|^2 + \|\ddot{w}\|^2 - \|\dot{w} - (B_1 - Z C_2^T D_{21}) Z^{-1} \ddot{z}\|^2$$
$$+ \ddot{z}^T \left[Z^{-1} (A + B_1 F_1) + (A + B_1 F_1)^T \right] Z^{-1} \ddot{z}$$

(8)

Similar to the first Riccati matrix in (5), we define the second Riccati matrix as

$$Q_Z \equiv (A + B_1 F_1) Z + Z (A + B_1 F_1)^T + B_1 B_1^T - Z C_2^T C_2 Z + \gamma^{-2} Z F_2^T F_2 Z$$

(9)
If the two coupled Riccati inequalities

\[ Q_X \leq 0 \quad \text{and} \quad Q_Z \leq 0 \]  

(10)

have positive symmetric solutions \( X \) and \( Z \), then a new candidate Lyapunov functional

\[ U = V + \gamma^2 W \]

satisfies the inequality

\[ U \leq -\|z\|^2 + \gamma^2 \|w\|^2 \]

The above inequality is equivalent to \( \int_0^T \|z\|^2 \, dt \leq \gamma^2 \int_0^T \|w\|^2 \, dt, \forall T > 0, \forall w \in L_2(0,T), \) arriving at the preset control objective as specified in (2) Therefore, any observer-embedded \( L_2 \)-gain controller in (3) is represented as a triple

\[ (K, L, W_{\text{cal}}) = (-B_1^T X, Z C_2^T, \gamma^{-2} B_1 B_1^T X) \]

\[ (11) \]

THE CENTRAL OBSERVER-EMBEDDED \( L_2 \)-GAIN CONTROLLER

Referring to (5) and (9), the central \( L_2 \)-gain controller in (10) is governed by two Riccati equations, \( Q_X = 0 \) and \( Q_Z = 0 \), as

\[
\begin{align*}
A^T X + X A + C_1^T C_1 - X B_2 B_2^T X + \gamma^{-2} X B_1 B_1^T X &= 0, \\
(A + B_1 F_1) Z + Z (A + B_1 F_1)^T + B_1 B_1^T Z - Z C_2 C_2 Z + \gamma^{-2} Z F_2^T F_2 Z &= 0
\end{align*}
\]

(12) (13)

Defining the Hamiltonian matrix as

\[ H_X = \begin{bmatrix}
A & -B_2 B_2^T + \gamma^{-2} B_1 B_1^T \\
-C_1^T C_1 & -A^T
\end{bmatrix}, \]

(14)

we can solve the Riccati equation (12) Similarity of the matrices \( H_X \) and \( -H_X^T \) implies that \( H_X \) must have \( n \) stable eigenvalues and another \( n \) unstable eigenvalues provided that the \( H_X \) does not have purely imaginary eigenvalues A \((2n \times n)\) matrix is formed by stacking the stable column eigenvectors, and then partitioned as

\[ \begin{bmatrix} X_1 & X_2 \end{bmatrix} \]

where \( X_1, X_2 \in \mathbb{R}^{n \times n} \) If \( X_1 \) is nonsingular, then (12) is solved by setting \( X = X_2 X_1^{-1} \) From the above definition of \( X \), one can easily conclude that the image of \( \begin{bmatrix} I \\ X \end{bmatrix} \) is the invariant space of \( H_X \) This stable solution is denoted as

\( X = \text{Ruc}(H_X) \) Similarly, a Hamiltonian matrix defined as

\[ H_Z = \begin{bmatrix}
(A + B_1 F_1)^T & \gamma^{-2} F_2^T F_2 - C_2^T C_2 \\
-B_1 B_1^T & -(A + B_1 F_1)
\end{bmatrix} \]

(15)

can solve (13) \( Z \in \text{Ruc}(H_Z), Z > 0 \)

We define an \( H_X \)-dual Hamiltonian matrix as

\[ H_Y = \begin{bmatrix}
A^T & -B_2 B_2^T + \gamma^{-2} C_1^T C_1 - C_2^T C_2 \\
-B_1 B_1^T & -A
\end{bmatrix} \]

(16)

and let \( Y \in \text{Ruc}(H_Y), Y > 0 \), which corresponds to the dual Riccati equation (12)

\[ A Y + Y A^T + B_1 B_1^T Y - Y C_2 C_2^T Y + \gamma^{-2} Y C_1^T C_1 Y = 0 \]

Equivalently,

\[ Q_Y \equiv Y^{-1} A + A^T Y^{-1} + Y^{-1} B_1 B_1^T Y^{-1} - C_2 C_2^T + \gamma^{-2} C_1^T C_1 = 0 \]

(17)
A relationship between $H_Z$ and $H_Y$ can be found as

$$H_Z = \begin{bmatrix} I & -\gamma^{-2}X \\ 0 & I \end{bmatrix} H_Y \begin{bmatrix} I & -\gamma^{-2}X \\ 0 & I \end{bmatrix}^{-1},$$

(18)

where $X \in \text{Ric}(H_X)$, $X > 0$. Since the images of the operators $[I]_X$ and $[I]_Z$ are the stable eigenspaces of $H_Y$ and $H_Z$, respectively, it follows from (18) that

$$\text{Im} \left[ \begin{bmatrix} I \\ Z \end{bmatrix} \right] = \text{Im} \left[ \begin{bmatrix} I \\ Y \end{bmatrix} \right] \text{Im} \left[ \begin{bmatrix} I - \gamma^{-2}XY \\ Y \end{bmatrix} \right] = \text{Im} \left[ Y (I - \gamma^{-2}XY)^{-1} \right]$$

(19)

Therefore, given $X \in \text{Ric}(H_X)$, $X > 0$, we obtain the following condition

$$(Z \in \text{Ric}(H_Z), Z > 0) \Leftrightarrow (Y \in \text{Ric}(H_Y), Y > 0 \text{ and } \rho(XY) < \gamma^2),$$

(20)

where it is implied that

$$Z = Y (I - \gamma^{-2}XY)^{-1}$$

(21)

Based on (3) and (11), the central observer-based $L_2$-gain controller becomes

$$\dot{x} = (A + \gamma^{-2}B_1B_1^TX - B_2B_2^TX) \dot{x} + Y (I - \gamma^{-2}XY)^{-1} C_2^T (y - C_2 \dot{x}),$$

$$u = -B_2^TX \dot{x},$$

(22)

where $X \in \text{Ric}(H_X)$, $X > 0$, $Y \in \text{Ric}(H_Y)$, $Y > 0$, and $\rho(XY) < \gamma^2$.

This controller is identical to the DGKF $H_\infty$ controller [1]. This shows the following

The DGKF $H_\infty$ controller is an observer-embedded $L_2$-gain controller, and it can be realized from the $H_2$ optimal controller by incorporating a calibration term into its embedded Luenberger observer to obtain robust estimation.

**FEASIBLE OBSERVER-EMBEDDED $L_2$-GAIN CONTROLLERS**

We pose the following question: With the same order of the central controller (22), can all feasible observer-based $L_2$-gain controllers be solved by formulating the Riccati inequalities as $Q_X \leq 0$, $Q_Y \leq 0$, and $\rho(XY) \leq \gamma^2$?

The answer to the above question is negative. If the answer was positive, then the LMI version of $H_\infty$ controller synthesis [2] could be obtained by directly extending the central controller governed by Riccati inequalities (20). We derive the feasible observer-embedded $L_2$-gain controllers to clarify these issues and arrive at the correct answer.

Substitution of (21) into $Q_Z \leq 0$ in (9) yields

$$(Y^{-1}\gamma^{-2}X) (A + \gamma^{-2}B_1B_1^TX + (A + \gamma^{-2}B_1B_1^TX) (Y^{-1}\gamma^{-2}X)

- C_2^T C_2 + \gamma^{-2}XB_2B_2^TX + (Y^{-1} - \gamma^{-2}X) B_1B_1^T (Y^{-1} - \gamma^{-2}X) \leq 0

\Leftrightarrow

Y^{-1}A + \gamma^{-2}Y^{-1}B_1B_1^TX + \gamma^{-2}X - \gamma^{-4}XB_1B_1^TX + A^TY^{-1} - \gamma^{-2}A^T X

+ \gamma^{-2}XB_1B_1^TY^{-1} - \gamma^{-4}XB_1B_1^TX - C_2^T C_2 + \gamma^{-2}XB_2B_2^TX + Y^{-1}B_1B_1^TY^{-1}

- \gamma^{-2}Y^{-1}B_1B_1^TX - \gamma^{-2}XB_1B_1^TY^{-1} + \gamma^{-4}XB_1B_1^TX \leq 0

\Leftrightarrow

(Y^{-1}A + A^TY^{-1}X - C_2^T C_2 + \gamma^{-2}C_1^TC_1)

- \gamma^{-2} (XA + A^TX - XB_2B_2^TX + \gamma^{-2}XB_1B_1^TX + C_1^TC_1) \leq 0

\Leftrightarrow

Q_Y - \gamma^{-2}Q_X \leq 0,$$  

which is the correct answer to the above-posed problem.
The central controller implied in (23) is solved by setting $Q_X = 0$, $Q_Y - \gamma^{-2}Q_X = 0$, and $\rho(XY) \leq \gamma^2$, which is equivalent to $Q_X = 0$, $Q_Y = 0$, and $\rho(XY) \leq \gamma^2$, and that is identical to the DGKF $H_\infty$ control law [1]. Instead, by erroneously setting $(Q_X \leq 0$, $Q_Y \leq 0$, and $\rho(XY) \leq \gamma^2)$ and using to (4) and (8), one may arrive at

$$V + \gamma^2 W + \|z\|^2 - \gamma^2\|w\|^2 \leq x^T Q_X x + \dot x^T (\gamma^2 Q_Y - Q_X) \dot x \leq \dot x^T (-Q_X) \dot x$$  \hspace{1cm} (24)

It follows from (24) that the control objective of energy gain in (2) is not guaranteed because $Q_X \leq 0$, implying that there is no robust estimation. On the other hand, for the output feedback control (where the measurement does not contain full information of the plant state) of LTI systems, we would like to postulate the following two criteria

- There is no optimal control if there is no optimal observer embedded
- There is no robust control if there is no robust observer embedded

Therefore, the answer to the above-posed question is stated as the following theorem

**THEOREM 1** Solutions of feasible observer-embedded $L_2$-gain controllers, having the same order as that of the generalized plant model, are obtained from the Riccati inequalities

$$Q_X \leq 0, \quad Q_Y - \gamma^{-2}Q_X \leq 0, \quad \text{and} \quad \rho(XY) \leq \gamma^2 \tag{25}$$

Next, we normalize feasible observer-embedded $L_2$-gain controllers by replacing $X$ with $\gamma X$ in (12), and $Y$ with $\gamma Y$ in (17). The resulting formulation in terms of Riccati inequalities becomes

$$A^T X + X A + \gamma^{-1} C_1^T C_1 - \gamma X R_2 B_2^T X + \gamma^{-1} X R_1 B_1^T X \leq 0,$$

$$Y^{-1} A + A^T Y^{-1} + \gamma^{-1} Y^{-1} B_1^T Y^{-1} - C_2^T C_2 \leq A^T X + X A - \gamma X B_2^T B_2 X + \gamma^{-1} X B_1 B_1^T X,$$

and the triple representing the controller (3) becomes

$$\{K, L, W_{cal}\} = \{-\gamma B_2^T X, \gamma Y(I - XY)^{-1} C_2^T, \gamma^{-1} B_1 B_1^T X\} \tag{25}$$

With no loss of generality [3], the generalized plant model (1) can be normalized by setting $\gamma = 1$ and replacing $w$ by $\gamma^{-1}w$. We arrive at the final normalized formulation of feasible observer-embedded $L_2$-gain controllers from solutions of Riccati inequalities as

$$A^T X + X A + C_1^T C_1 - X B_2 B_2^T X + X B_1 B_1^T X \leq 0,$$

$$Y^{-1} A + A^T Y^{-1} + Y^{-1} B_1^T Y^{-1} - C_2^T C_2 \leq A^T X + X A - X B_2^T B_2 X + X B_1 B_1^T X, \tag{26}$$

and the triple representing the controller in (3) is

$$(K, L, W_{cal}) = (-B_2^T X, Y(I - XY)^{-1} C_2^T, B_1^T X) \tag{27}$$

To compare to the linear fractional transformation (LFT) parameterization of $H_\infty$ controllers, (26) can be formulated by $(Q, S)$ parameterization as follows

**COROLLARY 1 TO THEOREM 1** Any free pair of parameters $(Q, S)$, with $Q \geq 0$ and $S \geq 0$, such that

$$A^T X + X A + C_1^T C_1 - X B_2 B_2^T X + X B_1 B_1^T X + Q = 0,$$

$$Y^{-1} A + A^T Y^{-1} + Y^{-1} B_1^T Y^{-1} - C_2^T C_2 + S = 0,$$

$$X - Y^{-1} \leq 0,$$

$$Q - S \leq 0, \tag{28}$$

determines a feasible observer-embedded $L_2$-gain (normalized) controller
The trade-off between control and estimation in the synthesis is achieved by the matrices \( Q \) and \( S \) in the sense that control is enhanced by increasing \( Q \) and estimation is enhanced by increasing \( S \).

To utilize the efficient numerical procedures in Hamiltonian Matrix and Linear Matrix Inequalities, (26) can be relaxed as

\[
\begin{align*}
X^{-1} A^T + AX^{-1} + X^{-1} C_1^T C_1 X^{-1} - B_2 B_2^T + B_1 B_1^T &= 0, \\
Y^{-1} A + A^T Y^{-1} + Y^{-1} B_1 B_1^T Y^{-1} - C_2^T C_2 + C_1^T C_1 &\leq 0, \\
X - Y^{-1} &\leq 0,
\end{align*}
\]  

and the results are summarized below

**Corollary 2 to Theorem 1** Given a normalized performance level of energy gain (i.e., \( \gamma = 1 \)), if

\[
X \in \text{Ric} \left[ \begin{array}{cc}
A & -B_2 B_2^T + B_1 B_1^T \\
-C_1^T C_1 & -A^T
\end{array} \right],
\]

\[
\begin{bmatrix}
Y^{-1} A + A^T Y^{-1} - C_2^T C_2 & Y^{-1} B_1 & C_1^T \\
B_1^T Y^{-1} & -I & 0 \\
C_1 & 0 & -1
\end{bmatrix} \leq 0;
\]

\[
\begin{bmatrix}
Y^{-1} & I \\
I & X^{-1}
\end{bmatrix} \geq 0,
\]

then \((K, L, W_{\text{cal}}) \equiv (-B_2^T X, Y(I - XY)^{-1} C_2^T, B_1 B_1^T X)\) represents a feasible observer-embedded \( L_2 \)-gain controller.

The feasible controller in (30) contains a central control and a feasible estimation.

At the expense of more involved numerical computations, Theorem 1 can be reformulated in terms of two linearly coupled matrix inequalities

**Corollary 3 to Theorem 1** All feasible observer-embedded \( L_2 \)-gain controllers, with a normalized performance level of energy gain (i.e., \( \gamma = 1 \)), can be formulated as the following two linearly coupled matrix inequalities

\[
\begin{bmatrix}
X^{-1} A^T + AX^{-1} - B_2 B_2^T & X^{-1} C_1^T \\
C_1 X^{-1} & -I & 0 \\
B_1^T & 0 & -I
\end{bmatrix} \leq 0;
\]

and

\[
\begin{bmatrix}
Z^{-1} A_Z + A_Z^T Z^{-1} - C_2^T C_2 & Z^{-1} B_1 & F_2^T \\
B_1^T Z^{-1} & -I & 0 \\
F_2 & 0 & -I
\end{bmatrix} \leq 0,
\]

where \( A_Z = A + B_1 F_1, F_1 = B_1^T X, \) and \( F_2 = -B_2^T X, \) and the resulting observer-embedded \( L_2 \)-gain controller becomes

\[
\begin{align*}
\dot{x} &= (A + B_1 F_1 + B_2 F_2 - Z C_2^T C_2) \dot{x} + Z C_2^T y, \\
u &= -B_2^T X \dot{x}
\end{align*}
\]

**SUMMARY AND CONCLUSIONS**

This paper presents formulation of mixed Linear Matrix Inequalities, Hamiltonian Matrix, and Linear Parameterization to provide solutions to feasible observer-embedded \( L_2 \)-gain controllers that are capable of explicitly estimating plant states. Such observer-embedded \( L_2 \)-gain controller can be realized from the \( H_2 \) optimal controller by introducing a calibration term in the Luenberger observer to obtain robust state estimation. It is also shown that the DGKF \( H_\infty \) controller is the observer-embedded central \( L_2 \)-gain controller. Since the \( L_2 \)-gain control law is not restricted to LTI systems, the proposed approach is applicable to synthesis of robust linear parameter varying (LPV) systems [4].
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