ABSTRACT

Combustion instability is a serious problem limiting the operating envelope of present day gas turbine systems using a lean premixed combustion strategy. Gas turbine combustors employ swirl as a means for achieving fuel-air mixing as well as flame stabilization. However swirl flows are complex flows comprised of multiple shear layers as well as recirculation zones which makes them particularly susceptible to hydrodynamic instability. We perform a local stability analysis on a family of base flow model profiles characteristic of swirling flow that has undergone vortex breakdown as would be the case in a gas turbine combustor. A temporal analysis at azimuthal wavenumbers \( m = 0 \) and \( m = 1 \) reveals the presence of two unstable modes. A companion spatio-temporal analysis shows that the region in base flow parameter space for constant density flow, over which \( m = 1 \) mode with the lower oscillation frequency is absolutely unstable, is much larger that that for the corresponding \( m = 0 \) mode. This suggests that the dominant self-excited unstable behavior in a constant density flow is an asymmetric, \( m=1 \) mode. The presence of a density gradient within the inner shear layer of the flow profile causes the absolutely unstable region for the \( m = 1 \) to shrink which suggests a possible explanation for the suppression of the precessing vortex core in the presence of a flame.

NOMENCLATURE

Dimensional reference quantities

\( U_{z,ref} \) Maximum forward axial flow velocity at a given axial location
\( R_{\text{max}} \) Radius at which maximum axial velocity occurs
\( \bar{\rho}_u \) Unburnt gas density

Non-dimensional symbols

\( \bar{U}_z \) Base flow velocity in the axial direction
\( \bar{U}_\theta \) Base flow velocity in the azimuthal direction
\( S \) Local swirl number
\( r_f \) Radial location of the flame, normalized by \( R_{\text{max}} \)
\( \gamma \) Density ratio of burnt to unburnt gas, \( \bar{\rho}_b/\bar{\rho}_u \)
\( \beta \) reverse flow ratio, i.e. normalized magnitude of max. reverse flow velocity
\( \alpha \) Perturbation wave number
\( \alpha_r \) Real part of wavenumber, \( = Re(\alpha) \)
\( \alpha_i \) Spatial growth rate, \( = Im(\alpha) \)
1 Introduction

The combustion instability in lean premixed gas turbine combustors is caused by coupling between unsteady heat release fluctuations and the combustor acoustic field. This coupling can potentially result in amplification of acoustic pressure oscillations in the combustor. This can cause undesirable levels of pollutant emissions, structural damage, loss of performance etc. [1]. Unsteady heat release in lean premixed combustors is due to wrinkling and distortion of the premixed flame sheet by unsteady flow structures in the combustor flow field. These flow structures arise from various hydrodynamic instability mechanisms associated with the combustor flow field. Burners in modern gas turbine combustors use swirling flows as a means to achieve fuel-air mixing and flame stabilization [2]. Vortex breakdown in these types of flows results in a flow field that is comprised of multiple shear layers and a central recirculation bubble, that are susceptible to multiple instability mechanisms [3, 4]. These flow instability mechanisms result in inherently unsteady self-excited flow structures such as the Precessing Vortex Core (PVC) in these flows [5–8]. Therefore, insight into hydrodynamic instability mechanisms causing flow unsteadiness in swirl flows is necessary, in order to develop physically realistic and quantitatively accurate reduced order models for the prediction of combustion instability.

The dynamics of swirling flows are to a large extent governed by its swirl number, i.e. the ratio of the axial flux of tangential momentum and the axial flux of axial momentum. Hydrodynamic instability in swirling jet flows have been studied in the context of vortex breakdown. Various kinds of vortex breakdown behaviour have been observed in swirling flows, e.g. bubble vortex breakdown, spiral vortex breakdown, helical vortex breakdown etc. [9]. These breakdown modes primarily determine the mechanism of flame stabilization and fuel-air mixing in gas turbine combustors. However, swirl numbers in gas turbine combustors are nominally much higher than the critical swirl number at which vortex breakdown occurs. Thus, the mechanism leading to vortex breakdown state has been considered as a primary hydrodynamic instability mechanism and have been reported in several prior studies [9–16]. Practical gas turbine combustors operate at high swirl numbers that are much higher than the critical swirl number beyond which vortex breakdown occurs. Therefore, the unsteady PVC observed in these flows can be conceptualized as a secondary hydrodynamic instability associated with the post vortex breakdown flow. Instabilities in flows such as these can be classified as locally convective or locally absolute on the basis of the nature of the flow response to an impulsive perturbation at large subsequent times, i.e. $t \rightarrow \infty$ [17, 18]. Impulsive perturbation at a point in a convectively unstable (CU) flow, results in spatially growing disturbances that are convected away from the point of disturbance. As such, the flow returns to its former quiescent state at large times in the absence of continuous forcing. On the other hand, impulsive forcing of an absolutely unstable (AU) flow generates disturbances that grow both temporally and spatially. Thus, an AU flow acts as a self-excited oscillator with a well defined, characteristic frequency. AU and CU flows interact with the combustor acoustic field through the unsteady heat release that they generate [19].

Several prior studies have characterized AU/CU nature in swirling jets on the basis of local swirl number and the ratio of reverse to forward flow using canonical baseflow models [20–25]. Post vortex breakdown, swirling flows have shear layers generated by axial and azimuthal velocity gradients and a reverse flow close to centreline, which makes them a candidate for absolute instability. Further, these flows have instability mechanisms, apart from just shear layer instability, which can interact with each other and determine the final unsteady nature of the flow [3, 26].

Several theoretical and experimental studies have identified multiple instability mechanisms occurring in purely rotating and swirling jet flows [3, 26–29]. From these studies, the various instability mechanisms can be broadly classified into (1) Azimuthal shear layer instability (2) Axial shear layer instability, (3) Centrifugal instability and (4) Kelvin’s instability. The azimuthal and axial shear layer instability are equivalent to the classical Kelvin-Helmholtz type instability, due to the presence of shear layers in the tangential and axial velocity profiles. These two types of instability will be referred to as azimuthal KH and axial KH instability in the rest of the paper. The centrifugal and the Kelvin’s instabilities are due to centrifugal and Coriolis forces acting on rotating fluids respectively.

The centrifugal instability can be explained as follows. The net force on a fluid particle in a purely rotating flow is due to
the resultant of the local centrifugal force and the force generated by the local pressure gradient. Thus, when the fluid particle is displaced radially away from the axis of rotation, the imbalance between the centrifugal force and the force due to the pressure gradient at the displaced location determines whether the particle continues to move away or returns to the initial position [30]. Rayleigh has derived a stability criterion to identify centrifugal instability of rotating flows subjected to axisymmetric perturbations by taking into consideration the change in square of circulation of the purely rotating laminar base state (see ref. [30, 31]). Gallaire et al [32] have partially generalized this criterion for both axisymmetric and non axisymmetric perturbations by showing that purely rotating flows in the large axial wavenumber limit are centrifugally unstable, with the axisymmetric mode being the most centrifugally unstable.

Kelvin’s instability is due to the restoring action of the Coriolis force on a rotating fluid particles when displaced from their radial location. This restoring action of the Coriolis force in a pure rotating flow produces neutrally stable, inertial waves which appear in a stability analysis as a continuous spectrum within a well defined frequency range [33]. The interaction between these neutral modes and axial KH modes have been studied by Loiseleux et al [4] where Loiseleux et al has shown that a drastic change in behaviour occurs when these two modes resonate (i.e. when both of the modes are having the same frequency). The instability mechanisms observed in a non-reacting flows have been studied individually by several groups on various context, but the interaction of these modes in a reacting flow (which is what is seen in a typical swirl flow combustor) has not been reported as per the authors knowledge.

In the present study we first perform an inviscid local temporal stability analysis in order to identify unstable modes for the base flow profiles introduced by Oberleithner et al [34]. In order to identify the dominant instability mechanism at various axial ($\alpha$) and azimuthal wavenumber ($m$) settings, we analyze the source terms of the linearized vorticity transport equation generated by the unstable mode. These source terms can be identified as being related to each of the above instability mechanisms. Thus, from the relative dominance of one set of terms over the others allows us to determine which of the four mechanisms is responsible for driving the instability. Next, we include a density variation on the base flow to assess the effect that the presence of a flame would have on instability characteristics. The density variation introduces a fluctuating baroclinic torque which can constructively or destructively interact with other mechanisms and thereby, modify the instability characteristics of the flow. We also perform a local spatio-temporal analysis to identify the absolute/convective [19] nature of the unstable modes. We identify regions of absolute/convective instability in a parametric space spanned by local swirl number ($S$), defined as the ratio of the maximum azimuthal to maximum axial flow velocity and back flow ratio ($\beta$), defined as the ratio of the magnitude of the centrifugal axial velocity to the maximum axial velocity. We show that a density jump within the inner shear layer of the annular jet causes suppression of the absolute instability for the $m = \pm 1$ modes due to the effect the fluctuating baroclinic torque.

The rest of the paper is organized as follows. Section 2 outlines the mathematical formulation, baseflow model and numerical techniques used in the present study. Section 3 enumerates the “Linearized vorticity transport” equations used in the present study. Section 4 discusses the results obtained and section 5 concludes the paper with an outlook on future work.

2 Formulation

The hydrodynamic stability analysis in the present study is performed on a nominally steady base flow that is representative of an axisymmetric post vortex breakdown swirling jet as shown schematically in fig. 1. The analysis is performed in the low Mach number $M \rightarrow 0$ limit and the base flow is assumed to be locally parallel, axisymmetric and inviscid. The base flow density is allowed to vary spatially. We do this in order to capture the influence of a premixed flame on the stability characteristics. The density is allowed to vary spatially. We do this in order to capture the influence of a premixed flame on the stability characteristics. Thus, linearizing the Navier-Stokes equations about a general and nominally steady base flow yields (eg. see ref. [30]).

\[
\frac{\partial p'}{\partial t} + \bar{p}(r) \left( \frac{\partial u'_r}{\partial r} + \frac{u'_r}{r} + \frac{1}{r} \frac{\partial u'_\theta}{\partial \theta} + \frac{\partial u'_z}{\partial z} \right) + u'_r \frac{\partial \bar{p}}{\partial r} = \frac{\partial p'}{\partial r} \tag{1}
\]

\[
\bar{p}(r) \left( \frac{\partial u'_r}{\partial r} + \frac{\partial u'_\theta}{\partial \theta} + \frac{\partial u'_z}{\partial z} \right) + \frac{2 \bar{U}_0(r) u'_\theta}{r} = \frac{\partial p'}{\partial r} \tag{2}
\]

\[
\bar{p}(r) \left( \frac{\partial u'_r}{\partial r} + \frac{u'_r}{r} \frac{\partial \bar{U}_0(r)}{\partial r} + \frac{\partial \bar{U}_0(r)}{\partial \theta} + \frac{\partial u'_\theta}{\partial \theta} + \frac{\partial \bar{U}_z}{\partial z} \right) = - \frac{1}{r} \frac{\partial p'}{\partial \theta} \tag{3}
\]

\[
\bar{p}(r) \left( \frac{\partial u'_r}{\partial r} + u'_r \frac{\partial \bar{U}_z}{\partial r} + \frac{\partial u'_\theta}{\partial \theta} + \frac{\partial \bar{U}_0(r)}{\partial r} + \frac{\partial \bar{U}_z}{\partial z} \right) = \frac{\partial p'}{\partial z} \tag{4}
\]
\[
\frac{\partial u'_r}{\partial r} + \frac{u'_r}{r} + \frac{1}{\rho} \frac{\partial \rho'}{\partial \theta} + \frac{\partial u'_z}{\partial z} = 0
\]  
\text{(5)}

where the radial, azimuthal and the axial components of velocities in the cylindrical polar coordinate system are given by subscripts \( r, \theta \) and \( z \) respectively. The primed quantities represent small perturbations about their respective base flow values. The axial and tangential derivatives of the base flow quantities as well as the radial component of the base flow velocity (ie. \( \bar{U}_r \)) have been neglected in eqs. (1)-(5) (axi-symmetric, locally parallel base flow assumption). Thus the base flow quantities \( \bar{U}_r, \bar{U}_z, \bar{\rho} \) and \( \bar{\rho} \) are assumed to vary only in the radial direction. All lengths in eqs. 1-5 are non-dimensionalized with the length \( R_{\max} \) given by radial location of peak axial base flow velocity, \( \bar{U}_{r, \text{ref}} \). The latter is chosen as the velocity scale for non-dimensionalization. Pressure and density have been non-dimensionalized using their respective base flow values in the unburnt gas.

Next, the following boundary conditions are imposed at \( r \to \infty \),

\[
q' \to 0
\]  
\text{(6)}

where \( q' = \{p', u'_r, u'_\theta, u'_z, \rho'\} \). Kinematic compatibility conditions at the centerline \( (r = 0) \) proposed by Batchelor et al [35], are imposed as follows,

\[
\begin{align*}
\frac{\partial u'_r}{\partial r} = \frac{\partial p'}{\partial r} = \frac{\partial \rho'}{\partial r} &= 0 \quad \text{if } m = 0 \\
\end{align*}
\]  
\text{(7)}

\[
\begin{align*}
p' &= u'_z = p' = 0 \\
u'_r + \frac{\partial u'_\theta}{\partial \theta} &= 0 \\
\frac{\partial u'_r}{\partial r} &= 0 \quad \text{if } |m| = 1 \\
\end{align*}
\]  
\text{(8)}

\[
\begin{align*}
p' &= u'_r = u'_\theta = 0 \\
u'_z &= p' = 0 \quad \text{if } |m| > 1 \\
\end{align*}
\]  
\text{(9)}

Note that the kinematic compatibility conditions for \( |m| = 1 \) mode do not cause the radial and azimuthal components of velocity fluctuations to go to zero at the centerline. Therefore, these fluctuations can displace the axis of rotation of the central vortex core, thereby, causing it to precess. The unsteady flow field generated by PVC has been reported to have a helical mode structure (i.e. \( |m| = 1 \)) in several studies [5,6]. This suggests that the presence of unstable \( |m| = 1 \) modes eventually results in the formation of the PVC.

Next, the perturbed quantities in the eqs. 1-5 are expressed in the normal mode form as follows,

\[
q'(r, \theta, z, t) = \hat{q}(r)e^{i(\alpha z + m\theta - \omega t)}
\]  
\text{(10)}

Figure 1 schematically shows a typical time averaged flow field, observed in a swirl flow combustor. The swirler induces an

![Density profile Axial velocity profile Azimuthal velocity profile](image)

**Figure 1.** Schematic showing the variation of base flow velocity and density variation along the radial direction.
azimuthal component to the velocity vector. At high swirl numbers, the flow transitions into a vortex breakdown state which creates a central recirculation zone (CRZ) (see fig. 1). The two shear layers formed due to flow separating from the centerbody edge and the burner lip, are referred to as the Inner Shear Layer (ISL) and the Outer Shear Layer (OSL) respectively. In addition, there are two Azimuthal Shear Layers (ASL), arising from the radial variation of $U_\theta$ within and outside the vortex core as shown schematically in fig. 1.

We use the following base flow velocity model for the axial ($\bar{U}_z(r)$) and azimuthal ($\bar{U}_\theta(r)$) velocity components, Axial flow velocity profile ([34, 36]):

$$\bar{U}_z(r) = 4BF_1[1 - BF_1]$$

where quantity $B$ determines the amount of back flow at the center line through the following relation,

$$B = 0.5[1 + (1 + \beta)^{1/2}]$$

The parameter $\beta$, is the non-dimensional reverse flow velocity magnitude at the center line. Azimuthal flow velocity profile is given by [34]

$$\bar{U}_\theta(r) = 4SF_3[1 - F_4]$$

where, $S$ is the local swirl number which is defined as the ratio between the maximum azimuthal velocity to the maximum axial velocity. The functions, $F_1$, $F_2$, $F_3$ and $F_4$ are defined as follows,

$$F_j(r) = \frac{1}{1 + (e^{2bj} - 1)^{bj}}, j = 1, 2, 3$$

The parameters $N_1$ and $N_2$ in eq. 11 determine the thicknesses of the ISL and the OSL respectively. We assume that the ISL thickness and the OSL thickness (see Figure 1) are the same in this paper, i.e. $N_1 = N_2 = N$. The parameter $b_j$ is determined for a particular value of shear layer thickness by constraining the axial velocity to be maximum at $r = 1$. The shear layer parameters $N_3$ and $N_4$, along with the fitting parameters $b_3$ and $b_4$, are given by the values specified in Table 1.

<table>
<thead>
<tr>
<th>$N_3$</th>
<th>$N_4$</th>
<th>$b_3$</th>
<th>$b_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.18</td>
<td>0.73</td>
<td>0.51</td>
<td>0.27</td>
</tr>
</tbody>
</table>

Flames in swirl flow combustors operating at various conditions have been observed to be stabilized within the ISL, the OSL or both [8]. We examine the influence of ISL stabilized flames on the hydrodynamic stability characteristics of the flow in this paper. The premixed flame is modelled as a variation of density along the radial direction as follows

Density profile:

$$\hat{\rho}(r) = \left(\frac{1 + \gamma}{2}\right) + \left(\frac{1 - \gamma}{2}\right) \tanh \left(\frac{r - r_f}{N_f}\right)$$

where, $N_f$ is the normalized half flame thickness, and $r_f$ is the location of the inflection point in the density profile. Note that $N_f = 0.2$ is used for all the reacting flow cases presented in this paper. The density ratio across the flame is given by $\gamma = \rho_b/\rho_a$ (see fig. 2). Using these base flow profiles (eqs. 11-15) in eq. 1-9, yields an eigenvalue problem that cannot be solved in closed form. Therefore, we use a numerical pseudospectral method to solve the eigenvalue problem as discussed in the next subsection.

### 2.2 Numerical method

The physical space, $r \in [0, r_{max}]$, is mapped into the computational space $[-1, 1]$ using the mapping function suggested by Malik et al [37] as follows,

$$\frac{r}{r_c} = \frac{1 + y}{1 - y + \frac{2r_c}{r_{max}}}$$

The mapping parameter $r_c$ in eq. 16 distributes the grid points between 0 to $r_{max}$ in such a way that half of the grid points are placed in between 0 to $r_c$ and the other half between $r_c$ to $r_{max}$. We have set $r_c = 2$ and $r_{max} = 300$ for all results presented in this

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paper. These values were found sufficient to achieve convergence of eigenvectors in this study.

Equations 1-5 written in terms of \( \hat{q} \) are transformed into computational space using the mapping function in eq.16. Next, these equations are discretized in computational space using the well known pseudospectral collocation method based on cardinal functions [38]. This yields a discrete form of the eigenvalue problem given by eqs. 1-5, which can be written symbolically as the following generalized eigenvalue problem,

\[
A \hat{q}_d = \omega B \hat{q}_d
\]  

(17)

The \( \hat{q}_d \) in equation 17 is the discrete equivalent of the vector \( \hat{q} = \{ \hat{\rho}, \hat{\alpha}_r, \hat{\beta}_r, \hat{\theta}, \hat{\rho}_z, \hat{\omega}_z, \hat{\alpha}_\theta, \hat{\beta}_\theta, \hat{\psi} \} \), evaluated at the collocation points. The matrices \( A \) and \( B \) are functions of axial wavenumber \( \alpha \) and azimuthal wavenumber \( m \). Boundary conditions are imposed by replacing the rows in the matrix \( A \) and \( B \) with expressions corresponding to the \( r = 0 \) and \( r = 300 \) collocation points with eqs. 6 - 9. We solve this the discrete eigenvalue problem (eq. 17) using the MATLAB eig function. Converged solutions for eigenvalues and eigenvectors, \( \hat{q}_d \), are obtained by using 100 collocation points for the constant density cases and 150 collocation points for the variable density cases presented in this paper.

The absolute/convective nature of flow instability modes is determined by the nature of response of the flow to an impulsive forcing. This is determined by finding \(( \omega_r, \alpha_r) \) such that \( d\omega / d\alpha = 0 \), i.e. the saddle points of the dispersion relation [17,18]. If \( Im(\omega_r) > 0 \) and \( Im(\alpha_r) < 0 \), the flow is absolutely unstable (AU). On the other hand if \( Im(\omega_r) < 0 \) and \( Im(\alpha_r) < 0 \) the flow is convectively unstable (CU). We adopt the algorithm proposed by Deissler [39] to find \(( \omega_r, \alpha_r) \) using eq. 17. Thus for given values of \( N, \gamma \) and \( r_f \), the values of \( \beta \) and \( S \) where the flow transitions from being \( AU \) to \( CU \) is given by \( \hat{\omega}_r, \hat{\beta}, \hat{S} = 0 \). This equation is solved for different values of \( S \) by using newton iteration along with the saddle point search algorithm. The newton iterations are continued until the residual is less than \( 10^{-6} \) for all results presented in this study.

3 Linearized vorticity transport equation

We use the linearized vorticity transport equation to gain insight into the relative influence of various instability mechanisms on the characteristics of the unstable modes in this study. The vorticity transport equation, linearized about a steady base flow, can be written in the normal mode form as follows,

\[
\left( \frac{\partial \Omega_y'}{\partial t} + \frac{U_\theta}{r} \frac{\partial \Omega_y'}{\partial \theta} + U_z \frac{\partial \Omega_y'}{\partial z} \right) = - \frac{i \omega_r}{r} \frac{dU_z}{dr} + \frac{i \omega_r}{r} \frac{dU_\theta}{dr} + \frac{i \omega_r}{r} \frac{U_\theta}{r}
\]

(18)

\[
\left( \frac{\partial \Omega_\theta'}{\partial t} + \frac{U_\theta}{r} \frac{\partial \Omega_\theta'}{\partial \theta} + U_z \frac{\partial \Omega_\theta'}{\partial z} \right) = \frac{d^2U_z}{dr^2} - \frac{1}{r} \frac{dU_z}{dr} - i \omega_r \frac{dU_z}{dr} + \frac{i \omega_r}{r} \frac{dU_\theta}{dr}
\]

(19)

The source terms on the RHS of eqs. 18-20 represent contributions to the net generation rate of vorticity fluctuations from rearrangement of base flow vorticity by velocity disturbances, unsteady vortex stretching due to base flow velocity gradients and fluctuating baroclinic torque due to base flow density gradients. The contribution to the net generation rate of fluctuating vorticity from various fundamental instability mechanisms can be deduced from eqs. 18-20 as follows. Consider first a purely rotational flow \( (U_z = 0) \) and axisymmetric perturbations (ie. \( m = 0, \alpha > 0 \)). The only instability mechanism driving the flow unsteadiness is due to the centrifugal instability in this case. Hence the terms, \( (2), (3), (7), (9) \) and \( (11) \) can be identified as centrifugal instability contributions. Likewise if \( |m| > 0, \alpha = 0 \), only azimuthal KH instability mechanism drives flow unsteadiness through the source term \( (9) \). Thus the centrifugal and azimuthal KH instability mechanisms are coupled through term \( (9) \) which appears as a non-zero source term when only one or the other mechanism alone is present. This term is referred to as the coupling term in the present study. The relative importance of azimuthal KH and centrifugal instability due to this term can be identified from the base flow characteristics and the perturbation wave vector orientations with respect to the azimuthal shear layer. The baroclinic source terms in the above equations appear
when there are density gradients across a flame in the present study, the contributions from the baroclinic torque to the vorticity field is through the terms $\hat{\omega}_r$ and $\hat{\omega}_z$. The remaining source terms in eqs. 18-20 can be identified as the contribution to the vorticity field from the axial KH instability mechanism. Thus, in the vorticity generation rate budget results presented in the forthcoming section, the componentwise generation rate contributions, from any single instability mechanism, is the componentwise resultant of all unsteady source terms associated with the mechanism. So in the rest of the paper, componentwise sum of terms corresponding to centrifugal instability (ie. $\hat{\omega}_r$ and $\hat{\omega}_z$) is referred to as “centrifugal” in each component of the unsteady vorticity generation rate. Likewise componentwise sum of terms corresponding to azimuthal KH instability and axial KH instability will be referred to as “azimuthal KH” and “axial KH” respectively in each component of the unsteady vorticity generation rate. We compute source terms of the eqs. 18–20 using the eigenvectors corresponding to eigenvalues of interest in the present paper. Thus, the dominant mechanism causing instability for that eigenmode can be identified from spatial source term budgets.

4 Results

Figure 3a shows a typical eigenvalue spectrum, determined from temporal analysis ($\alpha = 2, m = 1, \beta = 0.1, S = 1.0, N = 4, \gamma = 1, N_3 = 4.18$ and $N_4 = 0.73$). This result is typical and is seen for other values of $\alpha$ and $m$ as well. The two physically relevant unstable eigenvalues are shown using ‘+’ and ‘x’ markers in fig. 3. The two unstable eigenvalues are due to the stretching and rearrangement of vortex tubes within the inner and outer shear layers induced by various instability mechanisms (see section 3). The eigenvalue corresponding to the marker ‘+’ is denoted as the low frequency (LF) mode as it always has a lower frequency than the other mode for all cases studied in this paper. Therefore, the eigenvalue corresponding to ‘x’ marker in fig. 3a is denoted as the high frequency (HF) mode. The cluster of eigenvalues close to the real axis is an artifact of the discrete representation of the continuous spectrum of the dispersion relation, as well as some spurious eigenvalues arising from the numerical discretization. Figure 3b shows the spatial variation of $|\hat{\omega}_r|$ eigenvector magnitude (normalized with the maximum of all eigenvectors) corresponding to the LF and HF unstable modes. Notice that the amplitude of these modes is large at locations within the two axial shear layers, suggesting that the instability of these modes is driven primarily by the KH instability caused by shear in the axial flow profiles. This fact can be understood from spatial budgets of fluctuating vorticity generation rate source terms. Figure 4 shows the spatial variation of source terms of $\Omega_0$ and $\Omega_1'$ components corresponding to a disturbance identical to the LF eigenmode in fig. 4a. The source terms of both the components have been decomposed into contributions from axial KH instability, azimuthal KH instability and centrifugal instability as discussed in section 3. All data shown in fig. 4 are normalized by the maximum magnitude of axial KH instability source term associated with the $\Omega_0'$ component. Figures 4a-b clearly shows that the axial KH instability source term in $\Omega_0'$ dominates over the source terms from all other instability mechanisms - implying that this is the dominant driving mechanism for the combination of parameters corresponding to the result in fig. 3.

We next present results concerning the absolute/convective nature of these unstable modes for the constant density case, as determined from the spatio-temporal analysis. The HF mode was found to be convectively unstable over the entire range of $S$, $\beta$ and $m$ considered in this study. Figure 5 shows the transition boundaries between flow profiles showing AU and CU behavior of the LF mode for $m=0$ and $m=1$, corresponding to values of $N = 3$ and 4 in each case ($\gamma = 1$). The region to the left of each of the boundaries corresponds to CU flow profiles.

Consider first the $m = 0$ mode. Figure 5 shows a large change in the location of the AU/CU transition boundary at high values of $S$ with decreasing $N$. This is because, for $m=0$, the
Figure 5. Boundary between absolutely unstable and convectively unstable behavior for various values of \( m \) and axial shear layer parameters \( N (\gamma = 1) \). The arrows marked AU and CU respectively point into regions of absolutely and convectively unstable flow.

Figure 6. Spatial budgets of \( \Omega'_0 \) source term, showing contributions from various instability mechanisms at the saddle point (a) \( S = 0.3 \) (b) \( S = 0.9 \) in fig. 5 \( (m = 0, \beta = 0.4, N = 3, \gamma = 1, N_3 = 4.18 \) and \( N_4 = 0.73 \)). The centrifugal instability mechanism becomes increasingly important at larger swirl numbers. This fact can be seen from fig. 6 which shows the spatial variation of fluctuating vorticity generation rate source terms for disturbance identical to the eigenmodes corresponding for \( (\omega_s, \alpha_s) \) at \( (\beta, S) = (0.4, 0.3) \) and \( (0.4, 0.9) \), i.e., points 'P' and 'S' in fig. 5. The contribution from centrifugal instability terms to the generation rate of the \( \Omega'_0 \) component is comparable to the corresponding contribution from the axial KH mechanism for \( S = 0.9 \) case (see fig. 6b), as opposed to the \( S = 0.3 \) case (see fig. 6a) where the Axial KH contribution is dominant. Thus, decreasing in the value of \( N \), causes the shear layers in the axial base flow velocity profile to weaken and hence, weakens the quantitative influence of the axial KH instability mechanism at all values of \( S \). Thus, the centrifugal instability mechanism becomes dominant at smaller values of \( S \) resulting in a larger change in the AU/CU boundary location for \( S > 0.2 \) when compared to \( S < 0.2 \) (see fig. 6). Note that contributions from the azimuthal KH mechanism do not exist for an axi-symmetric \( (m = 0) \) type disturbances. The same behaviour is seen for the \( \Omega'_2 \) component as well and is not shown in the interest of brevity.

Figure 7. Spatial budgets of fluctuating vorticity generation rate source terms, showing contributions from various instability mechanisms at the saddle point (a) \( \Omega'_0 \) (b) \( \Omega'_2 \) \((m = 1, \beta = 0.1, S = 0.5, N = 4, \gamma = 1, N_3 = 4.18 \) and \( N_4 = 0.73 \))

Consider next the \( m=1 \) case shown in fig. 5. The change in \( N \) has a minimal effect on the location of the AU/CU boundary. Further, the region of convective instability is very small when compared to that of \( m=0 \) and occurs at much smaller values of both \( \beta \) and \( S \). Figure 7 shows the contribution of each instability mechanism to \( \Omega'_0 \) and \( \Omega'_2 \) components at \( (\beta,S) = (0.1,0.5) \) in fig. 5 \( (m = 1, N = 4, \gamma = 1, N_3 = 4.18 \) and \( N_4 = 0.73 \)). Figure 7a shows that the axial KH instability mechanism dominates over all other instability mechanisms and hence axial KH instability determines the instability characteristics of \( m=1 \) mode. This plot is typical and has been found to have the same behaviour for other values of base flow parameters. Next, fig. 5 also shows that this mode can be expected to dominate the overall unsteady dynamics of the flow since it is AU over the entire region over which the \( m=0 \) low frequency mode is convectively unstable. This is a clear indication of the fact that the predominant self-excited behavior in a nominally axi-symmetric swirl flow would be expected to be a helical, \( m=1 \) instability. This fact is in qualitative agreement with recent experimental observations [40]. Next, we consider the influence of a density variation generated by a premixed flame on these absolute/convective instability transition boundaries.

Figure 8 shows the AU/CU transition boundaries for the density jump located within the ISL, i.e. \( r_f = 1.0, 0.5, \gamma = 0.3 \) for the \( m=0 \) low frequency mode \( (N=4) \). The boundary for the \( \gamma = 1 \) case is also shown for reference. The large change in the location of the boundary is due to the additional influence of fluctuating baroclinic torque on the net generation rate of fluctuating vorticity in the flow (see eq.19). The strong stabilization of the absolute instability for the \( r_f = 1.0 \) case can be understood as follows. Figure 9a which shows the spatial variation of the magnitude of the source terms due to rearrangement and stretching and the fluctuating baroclinic torque for the \( \Omega'_0 \) component due to the \( m=0 \) LF mode. This plot is typical and corresponds to \( (\beta,S) = (0.35,0.50) \) (point Q in fig.8). The corresponding phase difference between these two source terms is also shown.
on the vertical axis on the right. Clearly the baroclinic torque terms peaks close to where the resultant of all other source terms peak while being out of phase with the latter. Thus the baroclinic torque counteracts the production of vorticity by rearrangement and stretching causing the mode to become temporally stable (and hence CU) for the \((\omega_s, \alpha_s)\) values corresponding to point Q. The opposite result can be seen from fig. 9b which is a similar source term plot for \(r_f = 0.5\). For \(r_f = 0.5\) case, the fluctuating baroclinic torque peaks at a location (ie. \(r \approx 0.5\)) where the contribution from resultant of other source terms are negligible. Thus, in this case, the fluctuating baroclinic torque increases the net fluctuating vorticity generation rate causing the flow to become temporally unstable for the \((\omega_s, \alpha_s)\) values at Q and hence, absolutely unstable. Thus, it is clear that the density gradient can have a significant stabilizing or destabilizing impact on the absolute/convective nature of the instability corresponding to the \(m=0\) low frequency mode, depending on the location of the density gradient.

Similar results can be seen for the \(m=1\) mode from fig. 10 which shows the location of the AU-CU transition boundary for various values of \(\gamma\) and \(r_f\) (N=4). Also shown is the \(\gamma = 1\) case for comparison. The presence of the baroclinic torque term, results in the \(m=1\) low frequency mode becoming CU over a larger region of the \(\beta - S\) plane when compared to the constant density case. The reason for this can again be seen in fig. 11(a)-(b) which plots the spatial variation of the magnitude of the vorticity generation rate source terms, of \(\Omega_s^\rho\) and \(\Omega_s^\omega\). These plots correspond to \((\beta, S) = (0.3, 0.5)\) (point R in fig. 10) for \(\gamma = 0.3, r_f = 1.0\). Note that the baroclinic contribution is in phase with the net contribution from all other source terms resulting in the flow becoming AU (see fig. 11(a)). However, the opposite is true for the corresponding result shown in fig. 12(a)-(b) for \(\gamma = 0.3, r_f = 0.75\). Further, the result in fig. 10 shows that the region of absolute instability is greatly diminished due to the influence of baroclinic torque generated by the density variation. This suggests that the PVC, which is associated with a \(m = 1\) type instability, should be suppressed in the presence of a density gradient in the ISL which is in qualitative agreement with prior experimental observations [41].

5 Conclusion

This paper analyzes local convective/absolute instability transition for velocity and density profiles characteristic of pre-

Figure 8. Boundary between absolutely unstable and convectively unstable behavior for various locations of the base flow density gradient \((N = 4, \gamma = 0.3 \text{ and } m = 0)\). The boundary for \(\gamma = 1\) (thick black line) is also shown for reference.

Figure 9. Spatial budgets of fluctuating vorticity generation rate source terms for the \(\Omega_s^\rho\) component due to the \(m = 0\) impulse response mode at point Q (see fig. 8), from rearrangement, stretching and baroclinic mechanisms (a) \(r_f = 1.0\) (b) \(r_f = 0.5\) \((N = 4)\).

Figure 10. Boundary between absolutely unstable and convectively unstable behavior for different locations of the base flow density gradient and density change parameter \(\gamma\) \((N = 4 \text{ and } m = 1)\). The boundary for \(\gamma = 1\) (thick black line) is also shown for reference.
mixed swirl stabilized combustors. We do this via a spatio-temporal hydrodynamic stability analysis about a locally parallel base flow with spatially varying density (as would be generated by a premixed flame), in the inviscid and low Mach number limit. Flows in swirl stabilized combustors are inherently complex as they are comprised of multiple shear layers and recirculation zones. In addition to the presence of shear layer instabilities, swirl flows can also be destabilized by other instability mechanisms such as the centrifugal instability and Kelvin’s instability. These different mechanisms are in general coupled and manifest themselves with varying importance at different flow conditions.

The initial temporal analysis of the base flow profiles considered, revealed the presence of two unstable modes. Of these, only the mode with the lower oscillation frequency showed transition between absolute and convective behavior. Interestingly, this study revealed that the first asymmetric mode, corresponding to an azimuthal wavenumber, \( m = 1 \), showed a very large region of absolute instability behavior when compared to the symmetric \( m = 0 \) mode. This suggests that the eventual, self-excited, unsteady behavior of the flow would be dominated by an \( m = 1 \) instability - a fact borne out by several observations of helical Precessing Vortex Core (PVC) phenomena in constant density swirl flows.

The inclusion of a flame induced density gradient in the inner shear layer (ISL) resulted in a dramatic stabilization of the absolute instability for the \( m=1 \) mode due to the influence of baroclinic torque. Thus, the present analysis suggests a possible reason to explain the suppression of the PVC in the presence of a flame - a fact that has been borne out by recent experiments. However, the actual location of the AU-CU transition boundary in the parameter space considered in this study is a strong function of the density gradient magnitude as well as its disposition relative to the shear layers in the flow. The same fact is seen to be true for the symmetric, \( m = 0 \) mode as well.

**Appendix A: Linearized inviscid Navier-Stokes equations: Normal mode form**

Using eq. 10 in eqs. 1-9 yields the following system of equations,

\[
-i\alpha \hat{\rho} + \rho \left( \frac{im\hat{u}_\theta}{r} + i\alpha \hat{u}_r \right) + \frac{im\hat{U}_\theta \hat{\rho}}{r} + \hat{u}_r \frac{d\hat{\rho}}{dr} = 0
\]  
\[i\alpha \hat{U}_\theta(r) \hat{\rho} = 0 \quad (21)
\]

\[
\hat{\rho} \left( -i\alpha \hat{u}_r + \frac{im\hat{U}_\theta \hat{u}_r}{r} + i\alpha \hat{U}_r \hat{\rho} - \frac{2\hat{U}_\theta \hat{u}_r}{r} \right) - \frac{\hat{U}_\theta \hat{\rho}}{r} = -\frac{d\hat{\rho}}{dr}
\]  
\[\frac{\hat{U}_\theta \hat{\rho}}{r} = -\frac{d\hat{\rho}}{dr} \quad (22)
\]
\[ \hat{\rho} \left( -i\omega \hat{u}_0 + \hat{a}_r \frac{d\hat{U}_0}{dr} + \frac{im\hat{U}_0 \hat{u}_0}{r} \right) + i\alpha \hat{U}_c \hat{u}_0 + \frac{\hat{U}_0 \hat{u}_c}{r} = -\frac{im\hat{\rho}}{r} \]  
\[ (23) \]

\[ \hat{\rho} \left( -i\omega \hat{u}_c + \hat{a}_r \frac{d\hat{U}_c}{dr} + \frac{im\hat{U}_0 \hat{u}_c}{r} + i\alpha \hat{U}_c \hat{u}_c \right) = -i\alpha \hat{\rho} \]  
\[ (24) \]

Next, the boundary conditions in the normal mode form at 
\[ r \to \infty \] 
and at the centerline (\( r = 0 \)) are obtained using eq. 10,

\[ \hat{q} \to 0 \]  
\[ (26) \]

\[ \begin{aligned} 
\hat{u}_r &= \hat{u}_0 = 0 \\
\frac{d\hat{u}_c}{dr} &= \frac{d\rho}{dr} = \frac{d\phi}{dr} = 0 \end{aligned} \]  
\[ \text{if } m = 0 \]  
\[ (27) \]

\[ \begin{aligned} 
\hat{\rho} &= \hat{u}_c = \hat{\rho} = 0 \\
\hat{u}_r + im\hat{u}_0 &= 0 \end{aligned} \]  
\[ \text{if } |m| = 1 \]  
\[ (28) \]

\[ \frac{d\hat{u}_c}{dr} = 0 \]  
\[ \text{if } |m| > 1 \]  
\[ (29) \]

REFERENCES


